THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MMAT5220 Complex Analysis and Its Applications 2019-20
Homework 2
Due Date: 5th March 2020

Compulsory Part

1. Suppose that $f(z)$ is differentiable at z_0, where $z_0 = r_0 e^{i\theta_0} \neq 0$. Show that the derivative $f'(z_0)$ can be written as
$$f'(z_0) = e^{-i\theta_0}(u_r + iv_r)$$
or
$$f'(z_0) = \frac{-i}{z_0}(u_\theta + iv_\theta),$$
where all the partial derivatives are evaluated at (r_0, θ_0).

2. Consider the following function
$$f(z) = \begin{cases} (1+i)\frac{\text{Im}(z^2)}{|z|^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0. \end{cases}$$
   (a) Show that the Cauchy-Riemann equations are satisfied at $z = 0$.
   (b) Is $f(z)$ differentiable at $z = 0$?

3. Let γ be the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ in the counterclockwise direction. Evaluate the integral $\int_\gamma z^m \bar{z}^n dz$ for any $m, n \in \mathbb{Z}$.

4. Evaluate the integral $\int_\gamma z^2 dz$, if
   (a) γ is a straight line segment from $z = 2$ to $z = 2i$;
   (b) γ is the major arc of the circle $\{z \in \mathbb{C} : |z| = 2\}$ from $z = 2$ to $z = 2i$.

5. Let γ be the arc of the circle $\{z \in \mathbb{C} : |z| = 2\}$ from $z = 2$ to $z = 2i$ that lies in the first quadrant. Show that
$$\left| \int_\gamma \frac{dz}{z^2 - 1} \right| \leq \frac{\pi}{3}.$$

6. Let γ_R be the arc of the circle $\{z \in \mathbb{C} : |z| = R\}$ from $z = R$ to $z = -R$ that lies in the upper half plane, where $R > 1$. Show that
$$\left| \int_{\gamma_R} \frac{z^2}{z^6 + 1} dz \right| \leq \frac{\pi R^3}{R^6 - 1},$$
and hence show that
$$\lim_{R \to +\infty} \int_{\gamma_R} \frac{z^2}{z^6 + 1} dz = 0.$$

Optional Part

1. Find the domain over which the function

$$f(z) = f(x+iy) = \left|x^2 - y^2\right| + 2i\,|xy|$$

is analytic.

2. Suppose that $f(z)$ is analytic on a domain D, where D is symmetric with respect to the real axis. Show that $g(z) := \overline{f(\bar{z})}$ is a well-defined analytic function on D.

3. Let γ_R be the circle $\{z \in \mathbb{C} : |z| = R\}$ in the counterclockwise direction. Show that, for $R > 2$,

$$\left|\int_{\gamma_R} \frac{3z-1}{z^4+4z^2+3}dz\right| \le \frac{2\pi R(3R+1)}{(R^2-1)(R^2-3)}.$$

4. Let γ_R be the vertical line segment from R to $R + 4\pi i$, where $R > 0$. Show that

$$\left|\int_{\gamma_R} \frac{2e^z}{1+e^{3z}}dz\right| \le \frac{8\pi e^R}{e^{3R}-1}.$$

5. Does the function $f(z) = \dfrac{1}{z^2}$ defined on $\mathbb{C} \setminus \{0\}$ have an antiderivative?