MATH 2050A - HW 3 - Solutions

We would be using the following Lemma.

Lemma 0.1. *Let* (x_n) *be a sequence of real numbers and* $r \in \mathbb{R}$. *Suppose* (x_n) *converges and* $L := \lim x_n < r$. *Then* $x_n < r$ *for all sufficiently large* n.

Proof. Note that $r - L > 0$. Hence there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|x_n - L| < r - L$, that is, $-(r - L) < x_n - L < r - L$. In particular, $x_n < r$ for all $n \geq N$. $\square$

Remark. If $<$ is replaced by $>$, the result still holds by essentially the same arguement.

Lemma 0.2. *Let* $r > 1$. *Then* $\lim_n r^{-n} = 0$.

Proof. This fact appeared in Tutorial 2 Example 6; you can use it without proof. We prove it here again for your convenience. We make use of the Binomial Theorem, which is true as long as addition and multiplication are commutative. (That means the Binomial Theorem readily follows from the *Algebraic Axioms* of $\mathbb{R}$ as a field.)

Define $\alpha := r - 1 > 0$. Then $r = r - 1 + 1 = 1 + \alpha$. By the binomial theorem, for all $n \in \mathbb{N}$, $r^n = (1 + \alpha)^n = \sum_{k=0}^{n} \binom{n}{k} \alpha^k$. Since $\alpha \geq 0$, for any $n, m \in \mathbb{N}$ with $0 \leq m \leq n$ we have

$$(1 + \alpha)^n = \sum_{k=0}^{n} \binom{n}{k} \alpha^k \geq \binom{n}{m} \alpha^m$$

Let $\epsilon > 0$. By Archimidean Principle, take $N \in \mathbb{N}$ with $N > 1/\alpha\epsilon$. Then for all $n \geq N$

$$\frac{1}{r^n} = \frac{1}{(1+\alpha)^n} \leq \frac{1}{\binom{n}{1}\alpha^1} = \frac{1}{n\alpha} \leq \frac{1}{N\alpha} < \epsilon$$

By the $\epsilon - N$ definition, $\lim_n r^{-n} = 0$ $\square$

Remark. In fact using the same proof technique here, we can show that for all $k > 0$ (consider only $k \in \mathbb{Q}$ at this stage), we have $\lim_n n^k r^{-n} = 0$ if $r > 1$

Solutions

1 (P.69 Q9)**.** Let $y_n := \sqrt{n+1} - \sqrt{n}$ for all $n \in \mathbb{N}$. Show that $(\sqrt{n} y_n)$ converges and find the limit.

Solution. We claim that $\lim_n \sqrt{n} y_n = \frac{1}{2}$.
Let $\epsilon > 0$. By Archimedean Principle, take $N \in \mathbb{N}$ such that $\frac{1}{4N} < \epsilon$. Then we have for all $n \geq N$

$$\begin{aligned}\left|\sqrt{n} y_n - \frac{1}{2}\right| &= \left|\sqrt{n}(\sqrt{n+1} - \sqrt{n}) - \frac{1}{2}\right| = \left|\frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n}} - \frac{1}{2}\right| = \left|\frac{\sqrt{n} - \sqrt{n+1}}{2(\sqrt{n+1} + \sqrt{n})}\right| = \left|\frac{1}{2(\sqrt{n+1} + \sqrt{n})^2}\right| \\ &\leq \frac{1}{2(2\sqrt{n})^2} \leq \frac{1}{4n} \leq \frac{1}{4N} \leq \epsilon\end{aligned}$$

The second row has made use of the fact that $(\sqrt{n})$ is an increasing sequence, which follows from that for all real $x \geq 0$, $x^2 \leq 1$ if and only if $x \leq 1$ The claim follows by the $\epsilon - N$ definition.

2 (P.69 Q20)**.** Let (x_n) be a sequence of positive real numbers such that $L := \lim(x_n^{1/n}) < 1$.

i. Show that there exists a real number $r \in (0, 1)$ such that $x_n \in (0, r^n)$ for all sufficiently large $n \in \mathbb{N}$.

ii. Hence, show that $\lim x_n = 0$.

Solution.

i. Take $r := (1 + L)/2$. Then $L < r < 1$. Since $x_n > 0$ for all $n \in \mathbb{N}$, $x_n^{1/n} > 0$. Hence, by order-preserving property of limit (Proposition 2.9, Lect), we have $L = \lim(x_n^{1/n}) \geq 0$. Therefore, $0 < r < 1$.
Note that $\lim(x_n^{1/n}) = L < r$. By Lemma 0.1, we have $x_n^{1/n} < r$ for sufficiently large n.
Hence, $x_n < r^n$ for sufficiently large n as x_n are positive for all $n \in \mathbb{N}$. This last note also shows that $0 < x_n < r^n$ for sufficiently large n.

ii. Since $0 < r < 1$, $r^{-1} > 1$. By Lemma 0.2, $\lim_n r^n = 0$. Since we have $0 \leq x_n \leq r^n$ for sufficently large n, by the sandwich theorem (proposition 2.10, Lect), $\lim x_n$ exists and $\lim x_n = 0$.

3 (P.69 Q22)**.** Let (x_n) be a covergent sequence of real numbers and (y_n) be such that for all $\epsilon > 0$ there exists $M \in \mathbb{N}$ such that $|x_n - y_n| < \epsilon$ for all $n \geq M$. Does it follow that (y_n) is convergent?

Solution. Yes. Let $L := \lim x_n$. We claim that $\lim y_n = L$.
Let $\epsilon > 0$. Then there exists N, M such that

$$|x_n - L| < \frac{\epsilon}{2} \text{ as } n \geq N$$
$$|x_n - y_n| < \frac{\epsilon}{2} \text{ as } n \geq M$$

Therefore, when $n \geq \max\{N, M\}$, we have by triangle inequality,

$$|y_n - L| \leq |y_n - x_n + x_n - L| \leq |x_n - y_n| + |x_n - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

The claim follows by the $\epsilon - N$ definition.