General information

- Tutor: Zhou Feng; Email: zfeng@math.cuhk.edu.hk;
- Tutorial time and venue: Th. 11:30–12:15, LHC 101;
- Course webpage: `https://www.math.cuhk.edu.hk/course/2122/math4010`
- References:
  - **Lecture Notes** of Prof. Leung (available on course webpage)
  - A. Bowers and N.J. Kalton, An introductory course in functional analysis, Springer, (2014).
  - E. Kreyszig, Introductory functional analysis with applications, John Wiley & Thusns (1978).
  - S. Ovchinnikov, Functional analysis, Springer, (2018).
- Structure of tutorials:
  1. A quick recall of the lecture content in the previous week.
  2. Explain examples and solve problems.
  3. Q & A.
- All the suggestions and feedback are welcome. Any report of typos is appreciated.

Recall

A normed space is a vector space equipped with a norm (1. **non-degenerate** positivity 2. scaling property 3. triangle inequality). A Banach space is a **complete** normed space. Besides the definition by the convergence of Cauchy sequence, completeness can be characterized via series [LN, Prop. 1.11]. For convenience, we use LN to refer the Lecture Notes.

Every normed space has a unique completion to Banach space which is implicitly given by [LN, Prop. 1.15]. Every Banach space with Schauder basis is separable but the inverse is false (P. Enflo 1973) (countering to Hilbert spaces).

Normed & Banach spaces

Example 1. Recall $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. For every $x = (x_1, \ldots, x_n) \in \mathbb{K}^n$, define

$$\|x\|_\infty := \max_{1\le i\le n} |x_i| \text{ and } \|x\|_p := (\sum_{i=1}^{n} |x_i|^p)^{1/p}.$$

Then $\ell^n_\infty := (\mathbb{K}^n, \|\cdot\|_\infty)$ and $\ell^n_p := (\mathbb{K}^n, \|\cdot\|_p)$ are finite dimensional normed spaces.

Proof. It is easily checked that $\|\cdot\|_\infty$ and $\|\cdot\|_p$ are positive, non-degenerate, and satisfy the scaling property. The triangle inequalities follow from $\max_{1\le i\le n}|x_i+y_i| \le \max_{1\le i\le n}(|x_i|+|y_i|) \le \max_{1\le i\le n}|x_i| + \max_{1\le i\le n}|y_i|$ and Minkowski inequality. □

Example 2. For $1 \le p < \infty$, put

$$\ell^p := \Big\{(x(i))_{i=1}^\infty \in \mathbb{K}^{\mathbb{N}} \colon (\sum_{i=1}^\infty |x(i)|^p)^{1/p} < \infty\Big\}$$

Equip ℓ^p with the norm $\|x\|_p := (\sum_{i=1}^\infty |x(i)|^p)^{1/p}$ for $x \in \ell^p$. Then $(\ell^p, \|\cdot\|_p)$, conventionally denoted by ℓ^p, is a Banach space.

Proof. Let $(x_n)_{n=1}^\infty$ be a Cauchy sequence in ℓ^p with respect to $\|\cdot\|_p$, i.e., $\forall\,\varepsilon > 0$, $\exists\, N \in \mathbb{N}$ such that $\forall\, m, n \ge N$, $\|x_m - x_n\|_p \le \varepsilon$.

First, we find a candidate of the limit of $(x_n)_{n=1}^\infty$. Fix any $i \in \mathbb{N}$. Since $\forall\, m, n \in \mathbb{N}$, $|x_m(i) - x_n(i)| \le (\sum_{j=1}^\infty |x_m(j) - x_n(j)|^p)^{1/p} = \|x_m - x_n\|_p$, we have $(x_n(i))_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{K}$. By the completeness of $\mathbb{K}$, there exists $x(i) = \lim_{n\to\infty} x_n(i)$. Hence we can define $x = (x(i))_{i=1}^\infty$.

Next we check that x is indeed the limit of $(x_n)_{n=1}^\infty$ in ℓ^p. Fix any $K \in \mathbb{N}$. For $\forall\,\varepsilon > 0$, when m, n are large enough,

$$(\sum_{i=1}^K |x_m(i) - x_n(i)|^p)^{1/p} \le (\sum_{i=1}^\infty |x_m(i) - x_n(i)|^p)^{1/p} = \|x_m - x_n\|_p \le \varepsilon.$$

Since K is finite, letting $m \to \infty$, we have

$$(\sum_{i=1}^K |x(i) - x_n(i)|^p)^{1/p} \le \varepsilon.$$

Since K is arbitrary, we have

$$\|x - x_n\|_p = (\sum_{i=1}^\infty |x(i) - x_n(i)|^p)^{1/p} = \sup_{K\in\mathbb{N}} (\sum_{i=1}^K |x(i) - x_n(i)|^p)^{1/p} \le \varepsilon.$$

Hence $\|x\|_p \le \|x_n\|_p + \varepsilon < \infty$ and $x = \lim_{n\to\infty} x_n$ in ℓ^p. □

Observe that for any $i \in \mathbb{N}$, $|x(i)| \le \|x\|_p$ allows us to control the values on each index directly by controlling $\|\cdot\|_p$ in ℓ^p, which makes it easier for us to find the candidate of limit. This also happens for the sup-norm. However, the advantage of 'pointwise control' are not shared by $L^p, 1 \le p < \infty$ ('the space of p-th power integrable functions') with integral norms. We need to spend some extra efforts (i.e., Borel-Contelli) on searching for the candidate first, and then prove the completeness.

Remark. During the part about Banach spaces, an important family of Banach spaces is $\ell^p, 1 \le p \le \infty$ and c_0. It is one of the major themes of our tutorials to understand the properties, applications of important theorems, and relationships of ℓ^p and c_0.

Example 3 (Inclusion relationship). $\ell^p \subset \ell^q$ for $1 \le p \le q \le \infty$. $\overline{c_{00}} = c_0 \subset \ell_\infty$.

Proof. Let $n \in \mathbb{N}$ and $1 \leq p \leq q < \infty$. For any numbers $a_1, \ldots, a_n \geq 0$, it is easily checked, e.g., by directly differentiating w.r.t. the exponent (after taking logarithm) or using the concavity ($\implies$ 'subadditivity' of $x^\alpha, 0 \leq \alpha \leq 1$) in the tutorial, that

$$(\sum_{i=1}^{n} a_i^q)^{1/q} \leq (\sum_{i=1}^{n} a_i^p)^{1/p}.$$

Letting $n \to \infty$, we have $\ell^p \subset \ell^q$. Notice that $(\sum_{i=1}^{\infty} a_i^p)^{1/p} < \infty \implies \forall\, i \in \mathbb{N}, a_i \leq M < \infty$, thus $\ell^p \subset \ell^\infty$. The other statements are proved in lecture notes. □

Remark. For $L^p[0,1]$, the inclusion relationship is reversed. (Why? The difference of counting measure and Lebesgue measure matters.)

Example 4. Show that $C[0,1]$ is a Banach space under the sup-norm but it is not complete under the $\|\cdot\|_1$.

Proof. In Homework 1, we have proved $C[0,1]$ is a Banach space under sup-norm. The key to prove the continuity of limiting function is to apply the arguments to change the limit process via uniform convergence (see e.g. MATH2060).

(Check $\|\cdot\|_1$ is indeed a norm on $C[0,1]$.) To prove $C[0,1]$ is not complete under $\|\cdot\|_1$, we find a Cauchy sequence $(f_n)_{n=1}^{\infty} \in C[0,1]$ under $\|\cdot\|_1$ but does not converge in $C[0,1]$. For each $n \in \mathbb{N}$, define

$$f_n(x) := \begin{cases} 0 & , x \in [0, 1/2 - 1/n) \\ n(x + 1/n - 1/2) & , x \in [1/2 - 1/n, 1/2) \\ 1 & , x \in [1/2, 1]. \end{cases}$$

It is readily checked that $(f_n)_{n=1}^{\infty} \in C[0,1]$ is a Cauchy sequence under $\|\cdot\|_1$. Suppose there exists $f \in C[0,1]$ such that $f_n \xrightarrow{\|\cdot\|_1} f$. We will obtain a contradiction by showing $f = 0$ on $[0, 1/2)$ while $f = 1$ on $[1/2, 1]$, thus $f \notin C[0,1]$.

For any $x_0 \in [0, 1/2)$. If $f(x_0) \neq 0$, by continuity there exists $\varepsilon_0 > 0$ and $\delta > 0$ such that $|f(x)| > \varepsilon_0$ on $[x_0 - \delta, x_0 + \delta] \cap [0,1]$. Hence $\int_{[x_0-\delta, x_0+\delta] \cap [0,1]} |f(x)| dx > \delta\varepsilon_0$. However, since $f_n \xrightarrow{\|\cdot\|_1} f$, there exists $N \in \mathbb{N}$ such that $f_N = 0$ on $[x - \delta, x + \delta]$ and

$$\int_{[x_0-\delta, x_0+\delta] \cap [0,1]} |f(x)| dx = \int_{[x_0-\delta, x_0+\delta] \cap [0,1]} |f(x) - f_N(x)| dx \leq \|f - f_n\|_1 \leq \delta\varepsilon_0/4,$$

which is a contradiction, thus $f(x_0) = 0$. Hence $f = 0$ on $[0, 1/2)$. A similar argument shows that $f = 1$ on $[1/2, 1]$. Together we have f is not continuous at $1/2$, thus $f \notin C[0,1]$. □

Remark. Example 4 shows that equipping an infinite dimensional vector space with different norms may lead to different topologies (countering to the finite dimensional case). Recall the space of Riemann integrable functions $\mathcal{R}[0,1]$ (modulo a.e. vanishing functions) that is a little bit larger than $C[0,1]$, is still not complete under $\|\cdot\|_1$, which is one of the reasons why we are consider the Banach space L^1.