THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4010 Functional Analysis 2021-22 Term 1

Solution to Homework 3

1. Let $p \in (0,1)$. Define

$$\ell_p := \left\{ (x_k)_{k=1}^{\infty} \in \mathbb{C} \colon \sum_{k=1}^{\infty} |x_k|^p < \infty \right\}.$$

For $x = (x_k)_{k=1}^{\infty}$ and $y = (y_k)_{k=1}^{\infty}$ in ℓ_p, define the metric d by

$$d(x,y) = \sum_{k=1}^{\infty} |x_k - y_k|^p.$$

Then (ℓ_p, d) is a metric vector space. Let $(b_k)_{k=1}^{\infty}$ be a bounded sequence in $\mathbb{C}$. Show that

$$f(x) = \sum_{k=1}^{\infty} b_k x_k \quad \text{for } x = (x_k)_{k=1}^{\infty} \in \ell_p$$

is a continuous linear functional on the metric vector space (ℓ_p, d).

Proof. We begin with a useful fact about convex (concave) functions. Since $\phi(x) = x^p, 0 < p < 1$ is concave on $[0, +\infty)$ and $\phi(0) \geq 0$, then for $xy = 0$, $\phi(x+y) \leq \phi(x) + \phi(y)$ and for $x, y > 0$,

$$\phi(x) = \phi\left(\frac{x}{x+y} \cdot (x+y) + \frac{y}{x+y} \cdot 0\right) \geq \frac{x}{x+y}\phi(x+y) + \frac{y}{x+y}\phi(0) \geq \frac{x}{x+y}\phi(x+y);$$
$$\phi(y) = \phi\left(\frac{y}{x+y} \cdot (x+y) + \frac{x}{x+y} \cdot 0\right) \geq \frac{y}{x+y}\phi(x+y) + \frac{x}{x+y}\phi(0) \geq \frac{y}{x+y}\phi(x+y).$$

Combining the above inequalities gives $\phi(x+y) \leq \phi(x) + \phi(y)$ for $x, y \geq 0$ (subadditivity).

Then for $(x_k)_{k=1}^{\infty} \in \mathbb{C}$,

$$\sum_{k=1}^{n} |x_k| = \left(\sum_{k=1}^{n} |x_k|\right)^{p \cdot (1/p)} \leq (\sum_{k=1}^{n} |x_k|^p)^{1/p},$$

where the last inequality is by the subadditivity of x^p. Letting $n \to \infty$, by monotoneness and continuity we have

$$\sum_{k=1}^{\infty} |x_k| \leq (\sum_{k=1}^{\infty} |x_k|^p)^{1/p}. \tag{1}$$

Next we prove f is a continuous linear functional. Denote $b = (b_k)_{k=1}^{\infty}$. Then $\|b\|_{\infty} < \infty$.

(i) (well-defined) Let $x = (x_k)_{k=1}^{\infty} \in \ell_p$. Then $\sum_{k=1}^{\infty} |x_k|^p < \infty$, thus $\sum_{k=1}^{\infty} |x_k| < \infty$ by (1). Hence

$$|f(x)| = \left|\sum_{k=1}^{\infty} b_k x_k\right| \leq \sum_{k=1}^{\infty} |b_k x_k| \leq \|b\|_{\infty} \sum_{k=1}^{\infty} |x_k| < \infty.$$

(ii) (linear) For $\alpha \in \mathbb{C}$ and $x = (x_k)_{k=1}^{\infty}, y = (y_k)_{k=1}^{\infty} \in \ell_p$,

$$f(\alpha x + y) = \sum_{k=1}^{\infty} b_k(\alpha x_k + y_k) = \alpha \sum_{k=1}^{\infty} b_k x_k + \sum_{k=1}^{\infty} b_k y_k = \alpha f(x) + f(y).$$

(iii) (continuous) For any $x = (x_k)_{k=1}^{\infty}, y = (y_k)_{k=1}^{\infty} \in \ell_p$,

$$\begin{aligned} |f(x) - f(y)| &= \left| \sum_{k=1}^{\infty} b_k (x_k - y_k) \right| \\ &\le \sum_{k=1}^{\infty} |b_k||x_k - y_k| \\ &\le \|b\|_\infty \sum_{k=1}^{\infty} |x_k - y_k| \\ &\le \|b\|_\infty d(x,y)^{1/p}, \end{aligned}$$

where the last inequality follows from (1). Hence f is continuous at x.

□

2. Let $C[0,1]$ be the vector space of continuous functions on $[0,1]$. Define $\delta(x) = x(0)$ for $x \in C[0,1]$.

   (a) Show that δ is a bounded linear functional if $C[0,1]$ is endowed with the sup-norm. Find the norm of δ.

   (b) Show that δ is an unbounded linear functional if $C[0,1]$ is endowed with the norm

$$\|x\| = \int_0^1 |x(t)| dt.$$

Proof. For $\alpha \in \mathbb{C}$ and $x, y \in C[0,1]$, we have

$$\delta(\alpha x + y) = (\alpha x + y)(0) = \alpha x(0) + y(0) = \alpha \delta(x) + \delta(y).$$

Then δ is linear.

(a) For any $x \in C[0,1]$, we have $|\delta(x)| = |x(0)| \le \|x\|_\infty$, thus $\|\delta\| \le 1$.
Let $x_0 \equiv 1$ on $[0,1]$. Then $x_0 \in C[0,1]$ and $\|x_0\|_\infty = 1$. It follows from $|\delta(x_0)| = |x_0(0)| = 1$ that $\|\delta\| \ge 1$. Together we have $\|\delta\| = 1$.

(b) For $n \in \mathbb{N}$, define

$$x_n(t) = \begin{cases} n - n^2 t/2 & t \in [0, 2/n]; \\ 0 & t \in (2/n, 1]. \end{cases}$$

Then $x_n \in C[0,1]$ and $\|x_n\| = \int_0^1 |x_n(t)| dt = 1$. It follows from $|\delta(x_n)| = |x_n(0)| = n$ that $\|\delta\| \ge n$. Letting $n \to \infty$, we have δ is unbounded.

□

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