THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4010 Functional Analysis 2021-22 Term 1
Solution to Homework 2

1. Show that vectors (e_n), where e_n is the sequence whose n-th term is 1 and all other terms are zero,

$$e_1 = (1, 0, 0, \ldots),$$
$$e_2 = (0, 1, 0, \ldots),$$
$$\ldots$$

form a Schauder basis in ℓ^p for every $p \in [1, +\infty)$ and in the spaces c_0 and c_{00}.

Proof. Let $x = (x(i))_{i=1}^{\infty}$ be a sequence in $\mathbb{R}$ or $\mathbb{C}$. For every $n \in \mathbb{N}$, define $s_n = \sum_{i=1}^{n} x(i)e_i$. Then $s_n \in c_{00} \subset \ell^{1 \leq p < \infty} \subset c_0$ for all $n \in \mathbb{N}$. Note that $x - s_n = (0, \ldots, 0, x(n+1), \ldots)$.

Convergence of the series

    (a) If $x \in \ell^p$, $1 \leq p < \infty$, then $(\sum_{i=1}^{\infty} |x(i)|^p)^{1/p} = \|x\|_p < \infty$. For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $n \geq N$, $(\sum_{i=n+1}^{\infty} |x(i)|^p)^{1/p} \leq \varepsilon$, thus

$$\|x - s_n\|_p = \left(\sum_{i=n+1}^{\infty} |x(i)|^p \right)^{1/p} \leq \varepsilon.$$

    Hence s_n converges to x in $\| \cdot \|_p$, i.e., $x = \lim_{n \to \infty} s_n = \sum_{i=1}^{\infty} x(i)e_i$ in ℓ^p.

    (b) If $x \in c_{00}$ or c_0, then for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $n \geq N$, $\sup_{i \geq n+1} |x(i)| \leq \varepsilon$, thus

$$\|x - s_n\|_\infty = \sup_{i \geq n+1} |x(i)| \leq \varepsilon.$$

    Hence s_n converges to x in $\| \cdot \|_\infty$, i.e., $x = \lim_{n \to \infty} s_n = \sum_{i=1}^{\infty} x(i)e_i$ in c_{00} or c_0.

Uniqueness of the expansion

    Let $(\alpha(i))_{i=1}^{\infty}$ be a sequence of scalars such that $\sum_{i=1}^{\infty} \alpha(i)e_i = 0 \in \ell^p, 1 \leq p < \infty$ or c_{00} or c_0. It suffices prove that $\alpha(i) = 0$ for all $i \in \mathbb{N}$.

    Suppose on the contrary that there exists $n_0 \in \mathbb{N}$ such that $\alpha(n_0) \neq 0$. Let $\| \cdot \|$ denote $\| \cdot \|_\infty$ or $\| \cdot \|_p$. Since $\| \cdot \|_p \geq \| \cdot \|_\infty$, we have $\| \cdot \| \geq \| \cdot \|_\infty$. By the convergence of $\sum_{i=1}^{\infty} \alpha(i)e_i$ in $\| \cdot \|$, there exists $N \geq n_0$ such that $\| \sum_{i=N+1}^{\infty} \alpha(i)e_i \| < |\alpha(n_0)|/2$. It follows from the triangle inequality that

$$0 = \| \sum_{i=1}^{\infty} \alpha(i)e_i \| \geq \| \sum_{i=1}^{N} \alpha(i)e_i \| - \| \sum_{i=N+1}^{\infty} \alpha(i)e_i \|$$

$$\geq \| \sum_{i=1}^{N} \alpha(i)e_i \|_\infty - \frac{|\alpha(n_0)|}{2}$$

$$\geq |\alpha(n_0)| - \frac{|\alpha(n_0)|}{2} > 0,$$

    which is a contradiction. Hence $\alpha(n) = 0$ for all $n \in \mathbb{N}$. $\square$

2. Let $X = \{x \in C[0,1] \colon x(0) = 0\}$ with the sup-norm, and let f be a linear functional on X defined by

$$f(x) = \int_0^1 x(t)dt.$$

Show that $\|f\| = 1$.

Proof. Since $|f(x)| = \left| \int_0^1 x(t)dt \right| \leq \int_0^1 |x(t)|dt \leq \|x\|_\infty$, we have $\|f\| \leq 1$.

For any $\varepsilon > 0$ small, define

$$x_\varepsilon(t) = \begin{cases} \dfrac{t}{\varepsilon} & \text{if } t \in [0, \varepsilon], \\ 1 & \text{if } t \in (\varepsilon, 1]. \end{cases}$$

Then $x_\varepsilon \in X$ with $\|x_\varepsilon\|_\infty = 1$ and $|f(x_\varepsilon)| = |\int_0^1 x_\varepsilon(t)dt| = 1 - \varepsilon/2$. Hence $\|f\| \geq 1 - \varepsilon/2$. Letting $\varepsilon \to 0$, we have $\|f\| \geq 1$, thus $\|f\| = 1$. $\qquad\square$

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