THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4010 Functional Analysis 2021-22 Term 1
Homework 7
Deadline: 2021-12-06 Monday

Notice:
- All the assignments must be submitted before the deadline.
- Each assignment should include your name and student ID number.

1. Let S be a bounded sesquilinear form on $X \times Y$. Define

$$\|S\| \coloneqq \sup\{|S(x,y)| : \|x\| = 1,\ \|y\| = 1\}.$$

Show that

$$\|S\| = \sup\left\{\frac{|S(x,y)|}{\|x\|\|y\|} : x \in X \setminus \{0\},\ y \in Y \setminus \{0\}\right\}$$

and

$$|S(x,y)| \le \|S\|\,\|x\|\|y\|,$$

for all $x \in X$ and $y \in Y$.

2. Let $T\colon \ell^2 \to \ell^2$ be defined by

$$T\colon (x_1, \ldots, x_n, \ldots) \mapsto (x_1, \ldots, \frac{1}{n}x_n, \ldots).$$

Show that the range $\mathcal{R}(T)$ is not closed in ℓ^2.

3. Let T be a bounded operator on a complex Hilbert space H.

(a) Show that the operators

$$T_1 = \frac{1}{2}(T + T^*) \quad \text{and} \quad T_1 = \frac{1}{2i}(T - T^*)$$

are self-adjoint.

(b) Show that T is normal if and only if the operators T_1 and T_2 commute.

— THE END —