THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4010 Functional Analysis 2021-22 Term 1
Homework 4
Deadline: 2021-10-25 Monday

Notice:
- All the assignments must be submitted before the deadline.
- Each assignment should include your name and student ID number.

1. If X and Y are Banach spaces and $T_n\colon X \to Y,\ n = 1, 2, \ldots$ a sequence of bounded linear operators, show that the following statements are equivalent:

   (a) the sequence $(\|T_n\|)$ is bounded,

   (b) the sequence $(\|T_n x\|)$ is bounded for every $x \in X$,

   (c) the sequence $(|f(T_n x)|)$ is bounded for every $x \in X$ and every $f \in Y^*$.

2. Show that the space
$$Y = \{X \in C^1[0,1]\colon x(0) = 0\}$$
equipped with the sup-norm is not a Banach space (cf. the following lemma).

   Lemma. *The sequence*
$$x_n(t) = \sqrt{(t - \frac{1}{2})^2 + \frac{1}{n}}, \qquad t \in [0,1]$$
converges uniformly to the function $x(t) = |x - 1/2|$ *on* $[0,1]$.

— THE END —