THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4010 Functional Analysis 2021-22 Term 1
Homework 3
Deadline: 2021-10-11 Monday

Notice:
- All the assignments must be submitted before the deadline.
- Each assignment should include your name and student ID number.

1. Let $p \in (0,1)$. Define
$$\ell_p := \left\{ (x_k)_{k=1}^{\infty} \in \mathbb{C} \colon \sum_{k=1}^{\infty} |x_k|^p < \infty \right\}.$$
For $x = (x_k)_{k=1}^{\infty}$ and $y = (y_k)_{k=1}^{\infty}$ in ℓ_p, define the metric d by
$$d(x,y) = \sum_{k=1}^{\infty} |x_k - y_k|^p.$$
Then (ℓ_p, d) is a metric vector space. Let $(b_k)_{k=1}^{\infty}$ be a bounded sequence in $\mathbb{C}$. Show that
$$f(x) = \sum_{k=1}^{\infty} b_k x_k \quad \text{for } x = (x_k)_{k=1}^{\infty} \in \ell_p$$
is a continuous linear functional on the metric vector space (ℓ_p, d).

2. Let $C[0,1]$ be the vector space of continuous functions on $[0,1]$. Define $\delta(x) = x(0)$ for $x \in C[0,1]$.
   (a) Show that δ is a bounded linear functional if $C[0,1]$ is endowed with the sup-norm. Find the norm of δ.
   (b) Show that δ is an unbounded linear functional if $C[0,1]$ is endowed with the norm
$$\|x\| = \int_0^1 |x(t)| dt.$$

— THE END —