1 Limit of Functions

Definition 1.1 (Cluster Point). Let $A \subset \mathbb{R}$ be a subset. Then $a \in \mathbb{R}$ is a cluster point (limit point) of A if and only if for all open interval I, we have $A \cap I \backslash \{a\} \neq \phi$. Equivalently, a is a sequential limit of some sequence (x_n) in A where $x_n \neq a$ for all $n \in \mathbb{N}$

Remark. It is not necessary for a cluster point to lie in the set. Consider $A := \{1/n : n \in \mathbb{N}\}$. Then 0 is a cluster point of A but $0 \notin A$.

Definition 1.2 (Functional Limits). Let $A \subset \mathbb{R}$ and $f : A \to \mathbb{R}$ be a function. Let $a \in \mathbb{R}$ be a cluster point of A. Then f has a limit at a if there exists $L \in \mathbb{R}$ such that for all $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in B(a, \delta)\backslash\{a\}$ (where $B(a, \delta) := (-\delta + a, a + \delta)$), we have $|f(x) - L| < \epsilon$. In fact it is easy to show that such L is unique and we write $\lim_{x\to a} f(x) := L$.

Example 1.3. *Show by using **only** the definition that*

i. The limit $\lim_{x\to 4} \frac{x^2 - 4}{x - 3} = 12$

ii. The limit $\lim_{x\to 1} \frac{x + 3}{x - 5} \neq 1$

iii. The limit $\lim_{x\to 0} \frac{1}{x}$ *does not exist.*

Solution.

i. Let $\epsilon > 0$. Take $\delta > 0$ such that $\delta < \min\{\epsilon/10, 1/2\}$. Now suppose $x \in B(4, \delta)$. Then we have $|x - 8| \leq |x - 4| + |4 - 8| \leq 4 + \delta \leq 5$ and $|x - 3| \geq |4 - 3| - |x - 4| \geq 1 - \delta \geq 1/2$. Hence,

$$\left|\frac{x^2 - 4}{x - 3} - 12\right| = \left|\frac{x^2 - 12x + 32}{x - 3}\right| = |x - 4|\left|\frac{x - 8}{x - 3}\right| \leq |x - 4|\frac{5}{1/2} \leq 10|x - 4| \leq 10 \cdot \epsilon/10 = \epsilon$$

ii. Let $\delta > 0$. Then take $x \neq 1$ such that $|x - 1| < \min\{\delta, 1\}$ Then we have $|x - 5| \leq |x - 1| + |1 - 5| = 1 + 4 = 5$

$$\left|\frac{x + 3}{x - 5} - 1\right| = \left|\frac{8}{x - 5}\right| \geq \frac{8}{|x - 5|} \geq \frac{8}{5}$$

iii. We first show that for all $\delta > 0$, there exists $x, y \in B(0, \delta)\backslash\{0\}$ such that $\left|\frac{1}{x} - \frac{1}{y}\right| \geq 1$. Let $\delta > 0$. Then take $\delta > x > 0$ such that $x < 1/2$. Take $y := -x$. Then it follows that we have $x, y \in B(0, \delta)\backslash\{0\}$ and

$$\left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{2}{x}\right| = \frac{2}{x} \geq 2 \cdot 2 = 4 \geq 1$$

Now suppose $L := \lim_{x\to 0} 1/x$ existed. Then there exists $\delta > 0$ such that $|1/x - L| < 1/2$ for all $x \in B(0, \delta)\backslash\{0\}$. It follows that **for all** $x, y \in B(0, \delta)\backslash\{0\}$, we have $|1/x - 1/y| \leq |1/x - L| + |1/y - L| < 1$. This contradicts the previous claim (why?). It must be the case that $\lim_{x\to 0} 1/x$ does not exist.

Quick Practice.

i. Prove the following limits using only the definition.

a) $\lim_{x\to -1} \frac{x + 5}{2x + 3} = 4$ *b)* $\lim_{x\to 3} \frac{x^2}{x - 2} \neq 1$ *c)* $\lim_{x\to 0} \frac{x}{|x|}$ *does not exist.*

d) $\lim_{x\to 0} \frac{x^2}{|x|} = 0$ *e)* $\lim_{x\to 4} \sqrt{x} \neq 1$ *f)* $\lim_{x\to 0} \frac{1}{x} (x > 0)$ *does not exist*

ii. Let $f : A \to \mathbb{R}$ *be a function and* $a \in \mathbb{R}$ *a cluster point of* A*. Suppose* $\lim_{x\to a} f(x) \neq L \in \mathbb{R}$*. Show that there exists* $\epsilon_0 > 0$ *and a sequence* (x_n) *in* A *such that* $\lim x_n = a$ *but* $|f(x_n) - L| \geq \epsilon_0$ *for all* $n \in \mathbb{N}$.

iii. Let $f : \mathbb{R} \to \mathbb{R}$ *be a function and* $a \in \mathbb{R}$ *.*

(a) Suppose $\lim_{x\to a} f(x)$ *exists. Show that for all* $\epsilon > 0$*, there exists* $\delta > 0$ *such that for all* $x, y \in B(\delta, a)\backslash\{a\}$*, we have* $|f(x) - f(y)| < \epsilon$.

(b) Prove that the converse is true. This is the Cauchy Criteria for functional limits.
*(Hint: You may use the **sequential criteria for limits**, which state that* $\lim_{x\to a} f(x) = L$ *if and only if every sequence* (x_n) *with* $x_n \to a$ *has the property that* $f(x_n) \to L$*, provided that a is a cluster point of the domain.)*

2 Other Notions of Functional Limits

Definition 2.1 (One-sided Limits)**.** Let $f : A \to \mathbb{R}$ be a function and $a \in \mathbb{R}$ a cluster point of A.

- We say that L is a left limit of f at a if for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $x \in \mathbb{R}$ with $x \in (a - \delta, a)$ we have $|f(x) - L| < \epsilon$. Left limit is unique if it exists; we write $\lim_{x \to a^-} f(x) := L$
- We say that R is a right limit of f at a if for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $x \in \mathbb{R}$ with $x \in (a, a + \delta)$ we have $|f(x) - L| < \epsilon$. Right limit is unique if it exists; we write $\lim_{x \to a^+} f(x) := R$.

Example 2.2. *Let* $f(x) := \dfrac{|x|}{x}$ *be defined on* $\mathbb{R}\backslash\{0\}$. *Find the left and right limits of* $f(x)$ *at* $x = 0$.

Solution. We first claim that $\lim_{x \to 0^-} = -1$. Let $\epsilon > 0$. Take $\delta := \epsilon$. Suppose $x \in (-\delta, 0)$. Then we have $f(x) = |x|/x = -x/x = -1$. Hence, $|f(x) - (-1)| = 0 < \epsilon$.
Next, we claim that $\lim_{x \to 0^+}$. Let $\epsilon > 0$. Take $\delta := \epsilon$. Suppose $x \in (0, \delta)$. Then we have $f(x) = |x|/x = x/x = 1$. Hence, $|f(x) - 1| = 0 < \epsilon$.

Definition 2.3 (Limit to Infinities)**.** Let $f : A \to \mathbb{R}$ be a function such that A is not bounded above. Then we say $L \in \mathbb{R}$ is a limit to $+\infty$ if for all $\epsilon > 0$, there exists $K > 0$ such that for all $x > K$, we have $|f(x) - L| < \epsilon$. The limit is in fact unique and we write $\lim_{x \to \infty} f(x) := L$

Example 2.4. *Show that* $\displaystyle\lim_{x \to \infty} \frac{2x^2 - 3}{x^2 - 1} = 2$.

Solution. Let $\epsilon > 0$. Take $K \in \mathbb{R}$ such that $2/K < \epsilon$ and $K \geq 2$ (why does such K exist?). Suppose $x \geq K$. We have

$$\left|\frac{2x^2 - 3}{x^2 - 1} - 2\right| = \left|\frac{2x^2 - 3 - 2x^2 + 2}{x^2 - 1}\right| = \left|\frac{-1}{x^2 - 1}\right| = \frac{1}{x^2 - 1} \leq \frac{1}{x^2 - x^2/2} = \frac{2}{x^2} < \epsilon$$

Quick Practice.

1. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a function. Formulate the definitions of the following.*

   a) $\lim_{x \to \infty} f(x) = L$ *b)* $\lim_{x \to -\infty} f(x) = L$ *c)* $\lim_{x \to \infty} f(x) = \infty$
   d) $\lim_{x \to -\infty} f(x) = -\infty$ *e)* $\lim_{x \to c^-} f(x) = \infty$ *f)* $\lim_{x \to c^+} f(x) = -\infty$

2. *Verify the following limits using only definitions.*

   a) $\displaystyle\lim_{x \to \infty} \frac{2x^3 + x + 1}{x^3 + 1} = 2$ *b)* $\displaystyle\lim_{x \to 0^+} \sqrt{x} = 0$ *c)* $\displaystyle\lim_{x \to 1^+} \frac{1}{x - 1} = \infty$
   d) $\displaystyle\lim_{x \to -\infty} \frac{2x^3 + x + 1}{x^3 + 1} = 2$ *e)* $\displaystyle\lim_{x \to \infty} \frac{\sin x}{x} = 0$ *f)* $\displaystyle\lim_{x \to 1^-} \frac{1}{x - 1} = -\infty$

3. *Let* $\lfloor x \rfloor$ *denotes the greatest integer not exceeding* x *for all* $x \in \mathbb{R}$.
   Define $f_R : \mathbb{R} \to \mathbb{R}$ *by* $f(c) := \lim_{x \to c^+} \lfloor x \rfloor$ *and* $f_L : \mathbb{R} \to \mathbb{R}$ *by* $f(c) := \lim_{x \to c^-} \lfloor x \rfloor$ *for all* $c \in \mathbb{R}$.
   Show that f_R, f_L *are well-defined and compute the function defined by* $f := f_R - f_L$.

3 Exercises

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function and $a \in \mathbb{R}$ a cluster point. Suppose $\lim_{x \to a} f(x) > 0$. Show that there exists $\delta > 0$ such that $f(x) > 0$ for all $x \in B(a, \delta) \cap A\backslash\{a\}$
2. Let $f : (a, b) \to \mathbb{R}$ be an increasing function, that is, $f(x) \leq f(y)$ whenever $x, y \in (a, b)$ and $x \leq y$. Suppose f is bounded above. Show that $\lim_{x \to b^-} f(x)$ exists.
   (Hint: You may want to apply the bounded monotone convergence theorem for sequences.)
3. Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function, that is, for all $x, y \in \mathbb{R}$ and $t \in [0, 1]$, we have
   $$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)$$
   (a) Let $x, y, z \in \mathbb{R}$ be such that $x < y < z$. Show that we have
   $$\frac{f(x) - f(y)}{x - y} \leq \frac{f(x) - f(z)}{x - z}$$
   (b) Show that for all $c \in \mathbb{R}$ the right limit $\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$ exists; in particular it does **not** diverge to infinities.
   (c) Show that $\lim_{x \to c} f(x) = f(c)$ for all $c \in \mathbb{R}$.

   (Hint: It is better for you to first think about the meaning (e.g. graphically) of a convex function.)