1 $\pm\infty$ as Limits

We will be mainly investigating sequences that **diverge** in this note.

Definition 1.1. Let (x_n) be a sequence of real numbers. Then we say that

- (x_n) diverges to $+\infty$, or $\lim x_n = +\infty$, if for all $M > 0$ there exists $N \in \mathbb{N}$ such that $x_n \geq M$ for all $n \geq N$ (that is $x_n \geq M$ eventually)
- (x_n) diverges to $-\infty$, or $\lim x_n = -\infty$, if for all $M > 0$ there exists $N \in \mathbb{N}$ such that $x_n \leq -M$ for all $n \geq N$ (that is $x_n \leq -M$ eventually)

Example 1.2. *Let $x_n := n/\sqrt{1+n}$ for all $n \in \mathbb{N}$. Show that $\lim x_n = \infty$.*

Solution. Let $M > 0$. Let $N \in \mathbb{N}$ such that $N > M$ by Archimedean Property. Suppose $n \geq 4N^2$. We have $x_n = \dfrac{n}{\sqrt{1+n}} \geq \dfrac{n}{\sqrt{3n+n}} = \dfrac{\sqrt{n}}{2} \geq \dfrac{\sqrt{4N^2}}{2} = N \geq M$. We conclude by definition.

Example 1.3 (Generalized Monotone Convergence)**.** *Let (x_n) be an increasing sequence. Show that it either converges or diverges to ∞.*

Solution. We split the question to two cases. First, suppose (x_n) is unbounded. Let $M > 0$. By unboundedness, there exists $N \in \mathbb{N}$ such that $x_N \geq M$. Note that (x_n) is increasing; therefore $x_n \geq x_N \geq M$ for all $n \geq N$. By definition, $\lim x_n = \infty$.
We leave the bounded part to the readers.

Example 1.4 (Generalized Compactness Theorem)**.** *Let (x_n) be a sequence. Show that it either has a subsequence that converges, a subsequence that diverges to ∞ or one that diverges to $-\infty$*

Solution. We leave it to the readers.

Example 1.5. *Let (x_n) be a sequence of positive real numbers. Show that $\lim 1/x_n = 0$ if and only if $\lim x_n = \infty$.*

Solution. ($\Rightarrow$). Let $M > 0$. Then there exists $N \in \mathbb{N}$ such that $1/x_n < 1/M$ for all $n \in \mathbb{N}$. This imples that $x_n \geq M$ for all $n \geq N$.
($\Leftarrow$). Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $x_n \geq 1/\epsilon$ for all $n \geq N$. This imples that $0 \leq 1/x_n \leq \epsilon$ for all $n \geq N$.

Remark. The requirement of positivity (or negativity) in this example is important. Consider $x_n := (-1)^n n$ for all $n \in \mathbb{N}$. Then clearly $\lim 1/x_n = 0$, but neither $\lim x_n = \infty$ nor $\lim x_n = -\infty$.

Example 1.6. *Let $x_n := 2^n$ for all $n \in \mathbb{N}$. Show that it is unbounded.*

Solution. Note that $\lim 1/x_n = (1/2)^n = 0$ by considering subsequences. Therefore $\lim x_n = \infty$. Note that (x_n) is increasing. By the generalized monotone convergence, it must be the case that (x_n) is unbounded.

Of course the above argument seems to be too much: the unboundedness of (2^n) can be shown using the binomial theorem. This is because we have for all $n \in \mathbb{N}$.

$$2^n = (1+1)^n = \binom{n}{0} + \binom{n}{1} + \cdots \binom{n}{n} \geq 1 + n$$

Quick Practice.

1. *Let (f_n) and (g_n) be two sequences of positive numbers. We write $f_n = O(g_n)$ if there exists $C > 0$ such that $f_n \leq Cg_n$ eventually. (This is the big-O notation.)*
   a). Show that if $\lim f_n = \infty$ then $\lim g_n = \infty$.
   b). Is the converse of part (a) true?
   c). Show that $f_n = O(g_n)$ if and only if $\overline{\lim} f_n/g_n < \infty$
2. *Let (f_n) and (g_n) be two sequences of positive numbers. We write $f_n = o(g_n)$ if for all $c > 0$ we have that $f_n \leq cg_n$ eventually. (This is the small-o notation.)*
   a). Show that $f_n = o(g_n)$ if and only if $\lim_n g_n/f_n = \infty$
   b). Suppose $f_n = o(g_n)$ and $g_n = o(h_n)$ where f, g, h are sequences of positive numbers. Show that $f_n = o(h_n)$.
   c). If $f_n = o(g_n)$ and $g_n = o(f_n)$, what can we say about the sequences?
   d). Let $x_n := 2^n$, $y_n := n!$ $z_n := n^n$ and $w_n := n^3$ for all $n \in \mathbb{N}$. Determine all possible asymptotic (big O, small o) among the sequences.

2 Some other Ways of Assigning Limits

Definition 2.1 (Cesáro Summability)**.** Let (x_n) be a sequence of real numbers. Define

$$c(x_n) = \frac{1}{n}(x_1 + \cdots + x_n)$$

for all $n \in \mathbb{N}$. Then we say that (x_n) is *Cesáro summable* if $\lim c(x_n)$ exists.

Remark. The terminology here may be a bit different from those in existing literature.

Example 2.2. *Consider* $x_n := (-1)^n$*. Then it is a divergent sequence. However* (x_n) *is Cesáro summable. In fact it is not hard to see that* $\lim c(x_n) = 0$.

Proposition 2.3. *Let* (x_n) *be a sequence of real numbers. Suppose* $\lim x_n = x \in \mathbb{N}$*. Then* (x_n) *is Cesáro summable and* $\lim c(x_n) = x$

Proof. It suffices to consider the case where $x = 0$ (why?). Suppose $\lim x_n = 0$. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $n \geq N$ would imply $|x_n| < \epsilon$. Furthermore, let $J \in \mathbb{N}$ such that $1/J < \epsilon / \sum_{i=1}^{N} |x_i|$ (we can safely suppose that $\sum_{i=1}^{N} |x_i| \neq 0$ (why?)).Now suppose $n \geq J, N$. Then we have

$$\begin{aligned} |c(x_n)| = \left| \frac{1}{n} \sum_{i=1}^{n} x_i \right| &= \left| \frac{1}{n} \sum_{i=1}^{N} x_i + \frac{1}{n} \sum_{i=N+1}^{n} x_i \right| \\ &\leq \frac{1}{n} \sum_{i=1}^{N} |x_i| + \frac{1}{n} \sum_{i=N+1}^{n} |x_i| \\ &\leq \frac{1}{J} \sum_{i=1}^{N} |x_i| + \frac{n-N}{n}\epsilon \leq \epsilon + \epsilon = 2\epsilon \end{aligned}$$

It follows that $\lim c(x_n) = 0 = \lim x_n$. □

Quick Practice.

1. *Let* (x_n) *be a sequence. Define* $c(x_n) := (x_1 + \cdots + x_n)/n$.
   a). *Show that if* $(c(x_n))$ *converge, then* $\lim x_n/n = 0$.
   b). *Construct a sequence such that* $\lim c(x_n)$ *does not exist.*
2. *Let* $A \subset \mathbb{N}$ *be a subset of natural numbers. Then for all* $n \in \mathbb{N}$*, we define*

   $$d_n(A) := \frac{|A \cap [1, n]|}{n} = \frac{\textit{number of elements in } A \cap [1, n]}{n}$$

   the probability of occurence of A *in first* n *natural numbers. Clearly* $d_n(A) \in [0, 1]$ *for all* $n \in \mathbb{N}$*. If* $(d_n(A))$ *converges, we say that* A *has natural density* $d(A) := \lim d_n(A)$.
   a). *Let* $E := \{2n : n \in \mathbb{N}\}$ *be the set of even numbers. Show that* $d(E) = 1/2$.
   b). *Let* $S := \{n^2 : n \in \mathbb{N}\}$ *be the set of square numbers. Show that* $d(S) = 0$.
   c). *Let* (x_n) *be a sequence of real numbers. We say that* (x_n) ***converges statistically*** *to* $x \in \mathbb{R}$ *if for all* $\epsilon > 0$ *that set*

   $$A_\epsilon := \{n \in \mathbb{N} : |x_n - x| \geq \epsilon\}$$

   has natural density $d(A_\epsilon) = 0$.
      i. *Show that if* (x_n) *converges in the ordinary sense to* $x \in \mathbb{R}$*, than* (x_n) *converges to* x *statistically.*
      ii. *Find an example of a sequence* (x_n) *that diverges in the ordinary sense but converges statistically.*