1 Cauchy Sequences

Definition 1.1. Let (x_n) be a sequence. Then we call (x_n) a Cauchy sequence if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$, we have

$$|x_n - x_m| < \epsilon$$

Remark.

- A Cauchy sequence can be thought of as a sequence whose terms are getting close to each other eventually.
- In some literature, the notion $\lim_{n,m\to\infty} |x_n - x_m| = 0$ is used to describe the definition of Cauchy sequence. We do not use this notation as it may cause confusion with the notion of iterated limits $\lim_{n\to\infty} \lim_{m\to\infty} x_{m,n}$ and $\lim_{m\to\infty} \lim_{n\to\infty} x_{m,n}$ for sequences (x_{mn}) with two indices.

Theorem 1.2 (Completeness of $\mathbb{R}$)**.** *Let (x_n) be a sequence of real numbers. Then (x_n) is a convergent sequence if and only if (x_n) is a Cauchy sequence.*

Remark. This result allows us to work with limits of sequence whose values are hard to compute!

1.1 Getting Used to the Definition

Example 1.3. *Show that $(x_n := 1/n)$ is a Cauchy sequence.*

Solution. Let $\epsilon > 0$ and pick $N \in \mathbb{N}$ such that $2/N < \epsilon$ by the Archimedean Property. Now suppose $n, m \geq N$ with $n, m \in \mathbb{N}$, we have

$$|x_n - x_m| = |1/n - 1/m| \leq |1/n| + |1/m| \leq 2/N < \epsilon$$

By definition, (x_n) is a Cauchy sequence.

Example 1.4. *Let (x_n) be a sequence of real numbers. Define $y_n := \sum_{k=1}^{n} |x_n|$ and $z_n := \sum_{k=1}^{n} x_n$. Show that if (y_n) converges, then (z_n) converges.*

Solution. It suffices to show that (z_n) is a Cauchy sequence. First observe that that for all $n, m \in \mathbb{N}$ with $n > m$, we have $z_n = x_1 + \cdots + x_n$ and $z_m = x_1 + \cdots + x_m$. Hence,

$$|z_n - z_m| = |x_{m+1} + \cdots + x_n| \leq |x_{m+1}| + \cdots + |x_n| = |y_n - y_m|$$

by the triangle inequality.
Now let $\epsilon > 0$. Since we know that (y_n) converges, in particular (y_n) is Cauchy, there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$ we have $|y_n - y_m| < \epsilon$. Hence by the above observation, when $n, m \in \mathbb{N}$ and without loss of generality $n > m$ (why can we do this?) we have

$$|z_n - z_m| \leq |y_n - y_m| < \epsilon$$

Therefore (z_n) is a Cauchy sequence and so converges.

Example 1.5. *Let (x_n) be a sequence of real numbers. Suppose for all $n \in \mathbb{N}$, we have $|x_n - x_{n+1}| < \frac{1}{2^n}$. Show that (x_n) is a Cauchy sequence.*

Solution. First notice that for all $n, m \in \mathbb{N}$ with $n > m$, we have

$$\begin{aligned} |x_n - x_m| &= |x_n - x_{n-1} + x_{n-1} - \cdots - x_{m+1} + x_{m+1} - x_m| \leq |x_n - x_{n-1}| + \cdots + |x_{m+1} - x_m| \\ &\leq \frac{1}{2^{n-1}} + \cdots + \frac{1}{2^m} \end{aligned}$$

by the triangle inequality. Notice that by the geometric progression identity,

$$\frac{1}{2^{n-1}} + \cdots + \frac{1}{2^m} = \frac{1}{2^m}\left(\frac{1}{2^{n-m-1}} + \cdots + 1\right) = \frac{1}{2^m}\frac{1 - (1/2)^{n-m}}{1 - 1/2} = \frac{(1/2)^m - (1/2)^n}{1 - 1/2} = \frac{1}{2}\left(\frac{1}{2^m} - \frac{1}{2^n}\right)$$

Now let $\epsilon > 0$. Note that the sequence $(1/2^n)$ is convergent and hence cauchy, so there exists $N \in \mathbb{N}$ such that $|1/2^n - 1/2^m| < \epsilon$. Hence, for all $n, m \geq N$ with $n > m$, we have by the above computation that $|x_n - x_m| \leq 1/2(1/2^m - 1/2^n) < \epsilon/2 < \epsilon$. By definition (x_n) is Cauchy.

1.2 Cauchy Sequence and Subsequences

Proposition 1.6 (Negation of the definition)**.** *Let (x_n) be a sequence. Then (x_n) is not Cauchy if and only if there exists $\epsilon_0 > 0$ and two subsequences (y_n) and (z_n) of (x_n) such that $|y_n - z_n| \geq \epsilon_0$*

Example 1.7. *Show that the sequence $(x_n := \cos(n\pi/2))$ does not converges.*

Solution. Consider the subsequences $(y_n := x_{4n} = \cos(2n\pi) = 1)$ and $(z_n := x_{4n+1} = \cos(\pi/2) = 0)$. Then for all $n \in \mathbb{N}$, we have $|y_n - z_n| = |1 - 0| = 1 \geq 1$. It follows that (x_n) is not a Cauchy sequence and so does not converge.

Example 1.8. *Show that the sequence $(x_n := \sqrt{n})$ does not converge.*

Solution. We show that (x_n) is not a Cauchy sequence. Consider the subsequences $(y_n := x_{n^2} = n)$ and $(z_n := x_{4n^2} = 2n)$. Then for all $n \in \mathbb{N}$, we have $|y_n - z_n| = |2n - n| = n \geq 1$. It follows that (x_n) is not a Cauchy sequence and so does not converge.

Example 1.9. *Let (x_n) be a Cauchy sequence. Show that there exists a subsequence (y_n) such that for all $n \in \mathbb{N}$, we have*

$$|y_{n+1} - y_n| \leq \frac{1}{2^n}$$

Solution. Consider the sequence $(\epsilon_n := 1/2^n)$. Then by definition of Cauchy sequence, there exists $N_1 \in \mathbb{N}$ such that $|x_n - x_m| < \epsilon_1$ for all $n, m \geq N_1$. Take $y_1 = x_{k_1}$ where $k_1 \geq N_1$.
Next, there exists $N_2 \in \mathbb{N}$ such that $|x_n - x_m| < \epsilon_2$ for all $n, m \geq N_1$. Then take $y_2 := x_{k_2}$ where $k_2 > N_2, k_1$.
Inductively, for all $\alpha \in \mathbb{N}$, there exists $N_\alpha \in \mathbb{N}$, such that $|x_n - x_m| < \epsilon_\alpha$. We then take $y_\alpha := x_{k_\alpha}$ where $k_\alpha > N_\alpha$ and $k_\alpha > k_{\alpha-1}, \cdots k_1$. Then by the construction it is clear that (y_n) is a subsequence of (x_n), furthermore

$$|y_{n+1} - y_n| = \left|x_{k_{n+1}} - x_{k_n}\right| < \epsilon_n = \frac{1}{2^n}$$

for all $n \in \mathbb{N}$ since $k_{n+1} \geq k_n \geq N_n$ for all $n \in \mathbb{N}$ by construction.

2 Exercise

1. For each of the following sequences (x_n), determine whether it is a Cauchy sequence **by definition or its negation**.

   a) $x_n := 1/n^2$ b) $x_n := 1 - (-1)^n$ c) $x_n := \sum_{k=1}^n \frac{1}{k(k+1)}$

   d) $x_n := \sum_{k=1}^n \frac{1}{k^2}$ e) $x_n := \sqrt{n} - \sqrt{n-1}$ f) $x_n := \sum_{k=1}^n \frac{1}{k}$

2. Following the textbook P.88 Definition 3.5.7, we call a sequence (x_n) contractive if there exists $C \in (0,1)$ such that for all $n \in \mathbb{N}$, we have $|x_{n+2} - x_{n+1}| \leq C|x_{n+1} - x_n|$.

   (a) Show that a contractive sequence must converge

   (b) Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function from the real domain. Suppose f is a contraction, that is, there exists $C \in (0,1)$ such that $|f(x) - f(y)| \leq C|x - y|$ for all $x, y \in \mathbb{R}$. Define $f_n := \underbrace{f(f(\cdots f}_{n \text{ times}}(0)))$. Show that (f_n) converges.

3. Find a Cauchy sequence (x_n) such that there exists no constant $C > 0$ with the property that

$$|x_n - x_{n+1}| \leq \frac{C}{2^n}$$

4. Let $r \in (0,1)$ and (a_n) be a bounded sequence of real numbers.

   (a) Define $y_n := \sum_{k=1}^n r^{k-1}$. Show that (y_n) converges.

   (b) Define $a := \limsup |a_n|^{1/n}$. Suppose $a < 1$. Show that there exists $0 \leq r < 1$ and $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$|a_n| \leq r^n$$

   (c) Define the nth partial sum $x_n := a_1 + \cdots + a_n = \sum_{k=1}^n a_i$ for (a_n). Now suppose $a := \limsup |a_n|^{1/n} < 1$. Show that (x_n) converges.