1 Introduction to the ϵ Argument

Proposition 1.1. *Let $a, b \in \mathbb{R}$. Then the following are equivalent:*

i. $a \leq b$

ii. (There exists $L > 0$) , for all $(L >)\epsilon > 0$, we have $a \leq b + \epsilon$

iii. (There exists $L > 0$) , for all $(L >)\epsilon > 0$, we have $a < b + \epsilon$

Remark. This simple observation shows that partial ordering is established as long as it is established for all numbers close enough to the numbers in concern.

Definition 1.2. Let $A \subset \mathbb{R}$ be a subset. Then we say

- A number $u \in \mathbb{R}$ is an *upper bound* of A if $a \leq u$ for all $a \in A$.
- A number $l \in \mathbb{R}$ is a *lower bound* of A if $l \leq a$ for all $a \in A$.
- An upper bound u of A is called a *supremum* if it is a lower bound to the set of upper bounds of A.
- A lower bound l of A is called an *infimum* if it is an upper bound to the set of lower bounds of A.

In fact, whenever a supremum (resp. infimum) exists, it is unique. We denote the supremum (resp. infimum) of a subset A, regardless of existence, by $\sup A$ (resp. $\inf A$).

Theorem 1.3 (ϵ-characterization of supremum)**.** *Let $A \subset \mathbb{R}$ be a subset and $s \in \mathbb{R}$. Then $s = \sup A$ if and only if*

i. s is an upper bound of A

ii. For all $\epsilon > 0$, there exists $a \in A$ such that $s - \epsilon < a$

iii. There exists $L > 0$ such that for all $L > \epsilon > 0$, there exists $a \in A$ such that $s - \epsilon < a$

Theorem 1.4 (ϵ-characterization of infimum)**.** *Let $A \subset \mathbb{R}$ be a subset and $s \in \mathbb{R}$. Then $s = \inf A$ if and only if*

i. s is a lower bound of A.

ii. For all $\epsilon > 0$, there exists $a \in A$ such that $a < s + \epsilon$.

iii. There exists $L > 0$ such that for all $L > \epsilon > 0$, there exists $a \in A$ such that $a < s + \epsilon$.

Remark. The ϵ-characterizations show that supremums (resp. infimums) are upper bounds (resp. lower bounds) that can *approximate* the set in concern.

Axiom 1.5 (Axiom of Completeness)**.** *Let $A \subset \mathbb{R}$ be a non-empty subset that is bounded above, that is, having an upper bound. Then its supremum $\sup A$ exists.*

Example 1.6. *Let $A := (0, 1)$. Find the supremum and infimum of A.*

Solution. First, it is clear that A is bounded above and so its supremum exists. Next, we claim $\sup A = 1$. It is clear that 1 is an upper bound. Now we proceed to show that 1 approximates A. Let $1/2 > \epsilon > 0$. Take $a := 1 - \epsilon/2$. Then $a \in (0, 1)$ and we clearly have $1 - \epsilon < a$. Therefore, by the ϵ-characterization of supremum, we have $\sup A = 1$.

Alternatively, we can view the argument this way: as 1 is an upper bound of A, we have $\sup A \leq 1$ by the definition of supremum. On the other hand, 1 approximates A and so $1 \leq \sup A$ (why?). Combining the two inequalities, we have $\sup A = 1$.

We leave it to the readers to show that $\inf A = 0$ using similar arguments.

Quick Practice. *For each of the following subsets X, determine and explain whether $\sup X$ and $\inf X$ exist. If yes, find them.*

a) $X = (0, 1]$

b) $X = [0, 1]$

c) $X = (-1, 0) \cup (1, 2]$

d) $X = \bigcup_{n=1}^{\infty} [n, n + 1)$

e) $X = \{|a - b| : a, b \in (0, 1)\}$

f) $X = \mathbb{Z}$

Remark. For part (e), $\sup X$ is called the diameter of $(0, 1)$

2 Countable Subsets in $\mathbb{R}$

Theorem 2.1 (Archimedean Property)**.** *Let $X = \mathbb{N} \subset \mathbb{R}$ be the set of natural numbers. Then X is not bounded above.*

Corollary 2.2 (ϵ- characterization of the Archimedean Property)**.** *Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $1/N < \epsilon$.*

Remark. This is basically saything that $\inf\{1/n : n \in \mathbb{N}\} = 0$

Example 2.3. *Let $X := \{1/n^2 : n \in \mathbb{N}\}$. Show that $\inf X = 0$.*

Solution. First 0 is a lower bound of X clearly. It remains to show that 0 approximates X by the $\epsilon-$characterization of infimum. Let $\epsilon > 0$. Then by the Archimediean Property, there exists $N \in \mathbb{N}$ such that $1/N < \epsilon$. Since $N \geq 1$ as $N \in \mathbb{N}$, we have $N \leq N^2$ (why?). Therefore, it follows that $1/N^2 \leq 1/N < \epsilon = 0 + \epsilon$ and so 0 approximates X. It follows that $\inf X = 0$.

Definition 2.4. Let $A \subset \mathbb{R}$ be a subset. We say that A is *dense* in $\mathbb{R}$ if for all $a < b \in \mathbb{R}$, we have $A \cap (a, b) \neq \phi$.

Theorem 2.5 (Density of $\mathbb{Q}$)**.** *The set of rational numbers $\mathbb{Q}$ is dense in $\mathbb{R}$*

Example 2.6. *Let $X := \mathbb{R}\backslash\mathbb{Q}$ be the set of irrational numbers. Show that X is dense in $\mathbb{R}$*

Solution. Let $a < b \in \mathbb{R}$. It suffices to find $x \in (a, b)$ that is irrational. Take $c := (a + b)/2$. Then $a < c < b$. By denseness of rational numbers, there exists $q_1, q_2 \in \mathbb{Q}$ such that $a < q_1 < c < q_2 < b$. Take $x := q_1 + (q_2 - q_1)/\sqrt{2}$. Then it is clear that x is irrational and $a < q_1 < x < q_2 < b$. Hence, $(a, b) \cap X \neq \phi$ and X is dense in $\mathbb{R}$

Quick Practice.

1. *For each of the following subsets X, determine and explain whether $\sup X$ and $\inf X$ exist. If yes, find them.*

a) $X = \mathbb{Q}$	*b)* $X = \{1/n^3 : n \in \mathbb{N}\}$	*c)* $X = \{(2n+3)/n^3 : n \in \mathbb{N}\}$
d) $X = \{1/q : q \in \mathbb{Q}\backslash\{0\}\}$	*e)* $X = \{q \in \mathbb{Q} : q^2 < 1\}$	*f)* $X = \{\lvert\sqrt{2} - q\rvert : q \in \mathbb{Q}\}$

2. *Let $A \subset \mathbb{R}$ be a subset. Show that A is a dense subset if and only if for all $\epsilon > 0$ and $r \in \mathbb{R}$, there exists $a \in A$ such that $|a - r| < \epsilon$. (This is the ϵ characterization of dense subsets).*

3 Exercises

1. Let $A \subset \mathbb{R}$ be a non-empty subset that is bounded below.
   a) Show that the subset $-A := \{-a : a \in A\}$ has a supremum.
   b) Show that $\inf A = -\sup(-A)$.
   c) Show that $\inf A = -\sup(-A)$ using an $\epsilon-$argument if you have not done so.
2. Let $A \subset \mathbb{R}$ be a non-empty bounded above subset. Suppose $u \in \mathbb{R}$ is an upper bound such that $u \in A$.
   a) Show that u is the unique upper bound of A that is in A.
   b) Show that $u = \sup A$.

   (We call such u the maximum of A and denote it by $\max A$)
3. Let $A \subset \mathbb{R}$ be a non-empty finite subset. Show that A has a maximum element, that is, an upper bound that lies in A.
4. We call $q \in \mathbb{Q}$ a dyadic fraction if $q = k/2^n$ for some $k \in \mathbb{Z}$ and $n \in \mathbb{N}$. Denote $\mathbb{D}$ the set of dyadic fractions.
   a) Show that $\{2^n : n \in \mathbb{N}\}$ is unbounded.
   b) Show that $\mathbb{D}$ is dense in $\mathbb{R}$. You may want to revisit the proof of the denseness of $\mathbb{Q}$.
5. Let $A \subset \mathbb{N}$ be an infinite subset. Show that A is not bounded above in $\mathbb{R}$. Is it true if $\mathbb{N}$ is replaced by $\mathbb{Q}$?