MATH 2058 - Revision Test 2 - Solutions

1 (15 marks)**.** Let (x_n) be a sequence in $\mathbb{R}$. We say that (x_n) diverges to $+\infty$ and write $\lim x_n = +\infty$ if for all $M > 0$, there exists $N \in \mathbb{N}$ such that $x_n \geq M$ for all $n \geq N$.

a) Let $x_n := n/\sqrt{n+1}$. Show that $\lim x_n = +\infty$ by definition.

b) Let (x_n) and (y_n) be sequences of postive numbers such that $\lim \dfrac{x_n}{y_n} = +\infty$. Show that if $\lim y_n = +\infty$ then $\lim x_n = +\infty$.

c) Is the converse of part (b) true? Prove your assertion.

Solution.

a. Let $M > 0$. Let $N \in \mathbb{N}$ such that $N > M$ by Archimedean Property. Suppose $n \geq 4N^2$. We have $x_n = \dfrac{n}{\sqrt{1+n}} \geq \dfrac{n}{\sqrt{3n+n}} = \dfrac{\sqrt{n}}{2} \geq \dfrac{\sqrt{4N^2}}{2} = N \geq M$. We conclude by definition.

b. Let $M > 0$. There there exists $N_1 \in \mathbb{N}$ such that $x_n/y_n \geq \sqrt{M} > 0$ for all $n \geq N_1$ since $\lim x_n/y_n = +$. There exists $N_2 \in \mathbb{N}$ such that $y_n \geq \sqrt{M} > 0$ for all $n \geq N_2$. Now take $N := \max\{N_1, N_2\}$. Then for all $n \geq N$, we have

$$x_n = y_n \cdot \frac{x_n}{y_n} \geq \sqrt{M} \cdot \sqrt{M} = M > 0$$

It follows from definition that $\lim x_n = +\infty$

c. No. We can take $x_n := n$ and $y_n := 1$ for all $n \in \mathbb{N}$. Then $\lim x_n = \lim x_n/y_n = \lim n = +\infty$ by the Archimedean Property. However, $\lim y_n = \lim 1 = 1$.

2 (15 marks)**.** Let (x_n) be a sequence. We denote $c(x_n) := \frac{1}{n}(x_1 + \cdots + x_n)$ for all $n \in \mathbb{N}$.

a) Find an example of a sequence (x_n) such that $\lim c(x_n)$ exists but $\lim x_n$ does not.

b) Show that in general if $\lim x_n$ exists then $\lim c(x_n)$ exists.

Solution.

a. Take $x_n := (-1)^n$ for all $n \in \mathbb{N}$. Observe when n is even, $c(x_n) = 1/n(-1+1-\cdots+1) = 0$.When n is odd, we have $c(x_n) = 1/n(-1+1-\cdots-1) = -1/n$. It follows that for all $n \in \mathbb{N}$, we have

$$|c(x_n)| \leq \frac{1}{n}$$

It follows from Squeeze Theorem that $\lim c(x_n) = 0$ as $\lim 1/n = 0$.

b. We first observe for all $x \in \mathbb{R}$ and sequences (x_n), we have $c(x_n - x) = c(x_n) - x$ for all $n \in \mathbb{N}$. Therefore, it suffices to consider the case where $x = 0$ (why?). Suppose $\lim x_n = 0$. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $n \geq N$ would imply $|x_n| < \epsilon$. Furthermore, let $J \in \mathbb{N}$ such that $1/J < \epsilon / \sum_{i=1}^{N} |x_i|$ (we can safely suppose that $\sum_{i=1}^{N} |x_i| \neq 0$ (why?)).Now suppose $n \geq J, N$. Then we have

$$\begin{aligned}
|c(x_n)| = \left| \frac{1}{n} \sum_{i=1}^{n} x_i \right| &= \left| \frac{1}{n} \sum_{i=1}^{N} x_i + \frac{1}{n} \sum_{i=N+1}^{n} x_i \right| \\
&\leq \frac{1}{n} \sum_{i=1}^{N} |x_i| + \frac{1}{n} \sum_{i=N+1}^{n} |x_i| \\
&\leq \frac{1}{J} \sum_{i=1}^{N} |x_i| + \frac{n-N}{n} \epsilon \leq \epsilon + \epsilon = 2\epsilon
\end{aligned}$$

It follows that $\lim c(x_n) = 0 = \lim x_n$.

Remark. The technique in Q2b is to split the sum into two parts with a small tail and a head that can be somehow controlled. This is a common technique when dealing with limits of sums.