MATH 2058 - Revision Test 2 - Solutions

1 (15 marks)**.** Use the ε-$\mathbb{N}$ definition to justify your answer if the following exists:

(i) (5 *points*) $\lim_n \dfrac{2n^2+n+1}{n^2+1}$.

(ii) (5 *points*) $\lim_n \dfrac{(-1)^n n^2+1}{2n^2-1}$.

(iii) (5 *points*) $\lim_n \dfrac{1}{n}\sin n$.

Solution.

i. We claim that $\lim \frac{2n^2+n+1}{n^2+1} = 2$. Let $\epsilon > 0$. By the Archimedean Property, there exists $N \in \mathbb{N}$ such that $1/N < \epsilon$. Suppose $n \geq N$. Then we have

$$\left|\frac{2n^2+n+1}{n^2+1} - 2\right| = \left|\frac{2n^2+n+1-2n^2-2}{n^2+1}\right| = \frac{n-1}{n^2+1} \leq \frac{n}{n^2+1} \leq \frac{n}{n^2} = \frac{1}{n} < \frac{1}{N} < \epsilon$$

By definition, $\lim \frac{2n^2+n+1}{n^2+1} = 0$.

ii. We claim that the limit does not exists. Write $x_n := \frac{(-1)^n n^2+1}{2n^2-1}$ for all $n \in \mathbb{N}$. We first show that for all $k \in \mathbb{N}$, there exists $m_k, n_k \geq k$ such that $|x_{m_k} - x_{n_k}| \geq \frac{1}{2}$. Suppose $k \in \mathbb{N}$. Take $n := 2k$ and $m := n+1$. Then we have

$$\begin{aligned}
|x_n - x_m| = \left|\frac{n^2+1}{2n^2-1} - \frac{-(n+1)^2+1}{2(n+1)^2-1}\right| &= \frac{(n^2+1)(2(n+1)^2-1) + ((n+1)^2-1)(2n^2-1)}{(2n^2-1)(2(n+1)^2-1)} \\
&\geq \frac{(n^2+1)(2(n+1)^2-1)}{(2n^2-1)(2(n+1)^2-1)} \\
&= \frac{n^2+1}{2n^2-1} \geq \frac{n^2}{2n^2} = \frac{1}{2}
\end{aligned}$$

in which we mainly used the fact that most terms in brackets are non-negative as $n \geq 1$. This proves the claim. Next, suppose it were true that $\lim x_n = L$ for some $L \in \mathbb{R}$. Then by $\epsilon - N$ definition, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|x_n - L| < \frac{1}{8}$. This implies that for all $n, m \geq N$, we have by the triangle inequality

$$|x_n - x_m| \leq |x_n - L| + |L - x_m| \leq \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

However, by the previous claim, there exist $i, j \geq N$ such that $|x_i - x_j| \geq \frac{1}{2} > \frac{1}{4}$. Contradiction arises. It must therefore be the case that (x_n) does not converge.

iii. We show that $\lim \frac{1}{n}\sin(n) = 0$. We mainly use the fact that $|\sin(x)| \leq 1$ for all $x \in \mathbb{R}$: let $\epsilon > 0$. Then by the Archimedean Property, there exists $N \in \mathbb{N}$ such that $1/N < \epsilon$. It follows that when $n \geq N$

$$\left|\frac{\sin n}{n}\right| = \frac{|\sin n|}{n} \leq \frac{1}{n} \leq \frac{1}{N} < \epsilon$$

By definition, $\lim \frac{1}{n}\sin n = 0$.

Comment. Most students were able to deal with (*i*) and (*iii*). Nonetheless, the performance for (*ii*) is poor: almost all students answered the question by considering subsequences **without using the $\epsilon - N$ definition**. The typical answer is to construct two subsequences and state the consistency of subsequential limits, which is not accepted unless one attempted to prove the consistency of subsequential limits using the $\epsilon - N$ definition.

2 (15 marks)**.** This question involves the compactness of real numbers.

(i) (7 *points*) Let $A = \{\frac{1}{n} : n = 1, 2, ...\} \cup \{0\}$. Use the definition of a compact set to show that the set A is compact.

(ii) (8 *points*) Show that the Cauchy criterion holds on every compact subset A of $\mathbb{R}$, that is, every Cauchy sequence in A has a limit in A.

Solution.

i. It suffices to show that every sequence in A has a convergent subsequence. Now let $x : \mathbb{N} \to A$ be a sequence in A (as a function of natural number). We split the question into two cases based on the range of x.

   First we suppose $x(\mathbb{N})$ is a finite set. Write $x(\mathbb{N}) = \{a_1, \cdots, a_k\}$ for some $k \in \mathbb{N}$. Then it must be the case that there exists $1 \leq i \leq k$ such that $x^{-1}(a_i)$ is infinite. Otherwise, we would have $\mathbb{N} = \bigcup_{i=1}^{k} x^{-1}(a_i)$ to be a finite union of finite subsets. This is clearly false. Now enumerate the infinite subset $x^{-1}(a_i)$ of $\mathbb{N}$ as $x^{-1}(a_i) = \{m_1, m_2, \cdots\}$ where $m_i < m_j$ whenenver $i < j$ and $i, j \in \mathbb{N}$. (This is possible due to the well-ordering property of $\mathbb{N}$). Then (x_{m_i}) is a constant subsequence of (x_n) which clearly converges.

   Next, we suppose $x(\mathbb{N})$ is infinite. We proceed with the following construction. Set $B_1 := \{n \in \mathbb{N} : \frac{1}{n} \in x(\mathbb{N})\}$, the collection of indices of the original sequences whose corresponding terms lie in the range of x, which is an infinite (hence non-empty) subset of $\mathbb{N}$. Take $b_1 := \min B_1$ (which exists by the well-ordering property of $\mathbb{N}$). Then pick $k(1) \in x^{-1}(\frac{1}{b_1})$. Set $B_2 := \{n \in \mathbb{N} : \frac{1}{n} \in x(\mathbb{N})\} \backslash \{n \in \mathbb{N} : \frac{1}{n} \in x(\mathbb{N} \cap [1, k(1)])\}$, the collection of indices of the original sequences whose corresponding terms cannot be in the initial parts of x, which is still an infinite (and hence non-empty) subset of $\mathbb{N}$ (why?). Take $b_2 := \min B_2$ and pick any $k(2) \in x^{-1}(\frac{1}{b_2})$.

   We then repeat the process and set $B_j := \{n \in \mathbb{N} : \frac{1}{n} \in x(\mathbb{N})\} \backslash \{n \in \mathbb{N} : \frac{1}{n} \in x(\mathbb{N} \cap [1, k(j-1)])\}$ (which is infinite and so non-empty), $b_j := \min B_j$ and $k(j) \in x^{-1}(\frac{1}{b_j})$ for all $j \geq 3$.
   We then obtain a strictly increasing map $k : \mathbb{N} \to \mathbb{N}$ by $j \mapsto k(j)$ (by the definition of B_j). Set $z_n := x \circ k(n)$. Then (z_n) is a subsequence with $z_n = \frac{1}{b_n}$. Note that in fact (b_n) is a strictly increasing sequence by minimality of b_i and the definition of B_i. It follows that (z_n) is a subsequence of $(\frac{1}{n})$ and so $\lim z_n = \lim \frac{1}{n} = 0 \in A$. It follows that (x_n) has a converging subsequence with a limit in A.

ii. Let (x_n) be a Cauchy sequence in A. Since A is compact, (x_n) has a convergent subsequence $(x_{k(n)})$ where k is strictly increasing with $a := \lim x_{k(n)} \in A$. It then suffices to show that $\lim x_n = a$. Let $\epsilon > 0$. Then there exists $N_1 \in \mathbb{N}$ such that $|x_{k(n)} - a| < \epsilon/2$ for all $n \geq N_1$. By Cauchiness of (x_n), there exists $N_2 \in \mathbb{N}$ such that $|x_n - x_m| \leq \epsilon/2$ for all $n, m \geq N_2$. Now suppose $j \geq N_1, N_2$. Then we have

$$|x_j - a| = |x_j - x_{k(j)} + x_{k(j)} - a| \leq |x_j - x_{k(j)}| + |x_{k(j)} - a| < \epsilon/2 + \epsilon/2 = \epsilon$$

   as $k(j) \geq j \geq N_1, N_2$. It follows that $\lim x_n = a \in A$. Therefore, the Cauchy criteria holds in A.

Remark. The proof of (i) can be used to show that for any convergent sequence (x_n) with $x := \lim x_n$, the set $\{x_n : n \in \mathbb{N}\} \cup \{x\}$ is compact.

Comment. To be updated.

3. We say that two sequences (x_n) and (x'_n) are equivalent if $\lim(x_n - x'_n) = 0$, write $(x_n) \sim (x'_n)$.

(i) (7 *points*) Suppose that $(x_n) \sim (x'_n)$. Show that if (x_n) is a Cauchy sequence, then so is (x'_n).

(ii) (8 *points*) Using the definition of a Cauchy sequence, show that if (x_n) and (y_n) are Cauchy sequences then so is the sequence $(|x_n - y_n|)$.

Solution.

i. Let $\epsilon > 0$. Since (x_n) is Cauchy, there exists $N_1 \in \mathbb{N}$ such that $|x_n - x_m| \leq \epsilon/3$ for all $n, m \geq N_1$. Since $\lim x_n - x'_n = 0$ as $(x_n) \sim (x'_n)$, it follows that there exists $N_2 \in N$ such that $|x_n - x'_n| < \epsilon/3$ for all $n \geq N_2$. Now suppose $n, m \geq \max\{N_1, N_2\}$. Then we have

$$|x'_n - x'_m| = |x'_n - x_n + x_n - x_m + x_m - x'_m| \leq |x'_n - x_n| + |x_n - x_m| + |x_m - x'_m| \leq \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$$

It follows that (x'_n) is a Cauchy sequence by definition.

ii. Write $z_n := |x_n - y_n|$. Let $\epsilon > 0$. Then there exist $N_1, N_2 \in \mathbb{N}$ such that

$$\begin{aligned} |x_n - x_m| &< \epsilon/2 && \text{if } n, m \geq N_1 \\ |y_n - y_m| &< \epsilon/2 && \text{if } n, m \geq N_2 \end{aligned}$$

by the Cauchiness of (x_n) and (y_n). It follows that if $n, m \geq \max\{N_1, N_2\}$, we have by the triangle inequality that

$$\begin{aligned} |z_n - z_m| &= ||x_n - y_n| - |x_m - y_m|| \\ &\leq |x_n - y_n - (x_m - y_m)| = |x_n - x_m + y_m - y_n| \\ &\leq |x_n - x_m| + |y_n - y_m| < \epsilon/2 + \epsilon/2 = \epsilon \end{aligned}$$

Therefore, (z_n) is a Cauchy sequence by definition.

Comment. The performance of this question is hugely disappointing. Given that the techniques involved include only restating the definition of a Cauchy sequence together with a very standard use of the triangle inequality, we expected this to be the easiest question. Nonetheless, nearly half of you failed in proving the results. If you find yourself to have lost marks in this question (not because of external factors like time constrains), you may want to reflect on your study method and to spend more time on the course. Please feel free to seek help from us or your classmates.