MATH 2058 - Home Project - Suggested Solutions (in brief)

1. (30 marks)

   (a) Let S be a countably infinite bounded subset of $\mathbb{R}$. Let D be the set of all limit points of S. Show that there exists a family of infinite subsets of $\mathbb{N}$, indexed by D, say $\mathcal{F} := \{N_\alpha : |N_\alpha| = \infty, \alpha \in D\}$, such that $N_\alpha \cap N_\beta$ is a finite set for all $\alpha \neq \beta \in D$.

   (b) Using part (a), show that there is an uncountable familiy of infinite subsets of $\mathbb{N}$, $\{N_i\}_{i\in I}$ where I uncountable, such that $N_\alpha \cap N_\beta$ is a finite set when $\alpha \neq \beta \in I$

   (c) Let $\mathcal{U}$ be a non-empty collection of subsets of $\mathbb{N}$ such that

      (i). For all $A, B \in \mathcal{U}$, we have $A \cap B \in \mathcal{U}$ and $A \cap B \neq \phi$

      (ii). For all $A \subset \mathbb{N}$ we have $A \in \mathcal{U}$ or $A^c \in \mathcal{U}$

   Let (x_n) be a bounded sequene of real numbers. Show that there exists L such that for all $\epsilon > 0$ such that $\{n :\in \mathbb{N} : |x_n - L| < \epsilon\} \in \mathcal{U}$. Is such L unique?

Solution.

(a) (cf. Tutorial 4 P. 2 Q3). We first construct a family of infinite subsets of S instead of $\mathbb{N}$ that has the given property.
Note that $x \in D$, a limit point of S, if and only if there exists a sequence (s_n) in $S\backslash\{x\}$ such that $\lim s_n = x$. Now for all $d \in D$, let $(s_{d,n})$ be a sequence in $S\backslash\{d\} \subset S$ such that $\lim_n s_{d,n} = d$. Define $N_d := \{s_{d,n}\} \subset S$ for all $d \in D$. Note that N_d are infinite subsets. Suppose not, that is, if N_d is a finite set for some d, then the sequence $(s_{d,n})$ is finite-valued. It follows that some value must attain infinitely many times, and so there is a subsequence converging to that value. However $s_{d,n} \neq d$ for all $n \in \mathbb{N}$. Therefore, that subsequence does not converge to d, which is not possible. It must be the case that N_d is infinite.
Next, we show that $\{N_d\}_{d\in D}$ satisfies that $N_d \cap N_c$ to be finite for all $d \neq c$.

Method 1: Direct Proof. Note that $c \neq d$, take $\epsilon := |c - d|/4$. It follows that there exists K_1, K_2 such that

$$|s_{d,n} - d| < \epsilon, n \geq K_1 \qquad\qquad |s_{c,n} - c| < \epsilon, n \geq K_2$$

Hence, for all $n \geq K_1$ and $m \geq K_2$, we have

$$\begin{aligned} |s_{d,n} - s_{c,m}| &= |s_{d,n} - d + d - c + c - s_{c,m}| \geq |d - c| - |s_{d,n} - d + c - s_{c,m}| \\ &\geq |c - d| - |s_{d,n} - d| - |c - s_{c,m}| \geq |c - d| - |c - d|/4 - |c - d|/4 \\ &= |c - d|/2 > 0 \end{aligned}$$

Hence, $s_{d,n} \neq s_{c,m}$ for all $n \geq K_1$ and $m \geq K_2$. It follows that at most K_2 terms in $(s_{d,n})_{n\geq K_1} \subset N_d$ are in $N_c \cap N_d$ while at most K_1 terms in $(s_{c,n})_{n\geq K_2}$ are in $N_c \cap N_d$ since the two tails share no common value. Hence $N_c \cap N_d$ is a finite set.

Method 2: Proof by Contradiction. Suppose not. Then $N_d \cap N_c$ is infinite for some $d \neq c$. We claim that there is a sequence, say (y_n) in $N_d \cap N_c$, that is both a subsequence of $(s_{d,n})$ and $(s_{c,n})$. The construction is similar to that in the solution to Q2a in midterm 1:

Enumerate $N_d \cap N_c = (\alpha_1, \cdots, \alpha_n, \cdots)$. Then take $j(1) := \min s_{d,n}^{-1}(\alpha_1)$ and $k(1) := \min s_{c,n}^{-1}(\alpha_1)$ (we consider the sequences as a function from natural numbers). Then we consider a tail of (α_n), say $(\alpha_n)_{n\geq \ell(1)}$ such that (α_n) does not contain any value in $(s_{d,n})_{n\leq j(1)}$ and $(s_{c,n})_{n\leq k(1)}$ (such tail exists since we are omitting only finitely many values). Next we take $j(2) := \min s_{d,n}^{-1}(\alpha_{n_1})$ and $k(2) := \min s_{c,n}^{-1}(\alpha_{n_1})$. We again consider a tail of $(\alpha_n)_{n\geq \ell(1)}$, say $(\alpha_n)_{n\geq \ell(2)}$ such that the tail does not contain any value in $(s_{d,n})_{n\leq j(2)}$ and $(s_{c,n})_{n\leq k(2)}$. Repeating the process, we have the strictly increasing sequences of natural numbers, $(\ell(n))$, $(j(n))$ and $(k(n))$ such that the sequence $(\alpha_{\ell(n)})$, which is a subsequence of (α_n), can be written as both $(s_{d,j(n)})$ and $(s_{c,k(n)})$. (In this brief solution, we leave it for the readers to verify the last statement in details.)

As $N_c \cap N_d$ is bounded, there exists a subsequence of the just-constructed sequence (y_n) that converges by B-W theorem. Since sub-sub sequences are sub-sequences, it follows that the subsequence converges to both the limits of $(s_{d,n})$ and $(s_{c,n})$, which are d, c respectively. However $c \neq d$ but limits are unique. Contradiction arises.
It follows that there exists a collection of subsets of S that satisfies the given property.

Lastly, since S is countably infinite, by considering a bijection from S to $\mathbb{N}$, we have a collection of subsets of $\mathbb{N}$ that satisfies the given property by considering the image of $\{N_d\}$ under the bijection.

(b) Take $S = \mathbb{Q} \cap [0,1]$. Then the set of limit points is $[0,1]$, which is uncountable. The collection of subsets constructed in (a) yield the required uncountable family.

(c) We first show that such L is unique. This can give hints to what L should be. Suppose not. Let L_1, L_2 be different numbers satisfying the property. Let $\epsilon > 0$. Then the set $S_i := \{n \in \mathbb{N} : |x_n - L_i| < \epsilon/2\} \in \mathcal{U}$ for $i = 1, 2$. By (i), it follows that $S_1 \cap S_2 \neq \phi$. Hence, there exists $n \in \mathbb{N}$ such that $|x_n - L_i| < \epsilon/2$ for $i = 1, 2$. It follows from triangle inequality that $|L_1 - L_2| < \epsilon$. Hence, $L_1 = L_2$ as ϵ is arbitrary. This shows the uniqueness of L.

We proceed to show the existence of L by an explicit construction. (Note that the uniqueness of L implies that we cannot pick arbitrary subsequential limits of (x_n) as a candidate.) We first prove the following property (iii) for the collection $\mathcal{U}$:

(iii). For all $A \in \mathcal{U}$, if $B \supset A$ then $B \in \mathcal{U}$.

We have the property because if $B \notin \mathcal{U}$ and $B \supset A \in \mathcal{U}$, then by (ii), we have $B^c \in \mathcal{U}$. Hence, by (i), we have $A \cap B^c = A \backslash B \in \mathcal{U}$. However, $A \cap B^c = \phi$ as $A \subset B$, which contradicts to the other part of (i).
We are now ready to perform the construction, which is analog to the bisection method. First suppose (x_n) are in $I_1 := [a_1, b_1]$ as (x_n) is bounded. Let c_1 be the midpoint of a_1, b_1. Now write $L_1 := [a_1, c_1]$ and $R_1 := (c_1, b_1]$ and so $I_1 = L_1 \sqcup R_1$. Now define

$$F_1 := \{n \in \mathbb{N} : x_n \in L_1\}, F_1^c = \{n \in \mathbb{N} : x_n \in R_1\}$$

By property (ii), either $F_1, F_1^c \in \mathcal{U}$. If $F_1 \in \mathcal{U}$, take $I_2 := L_1$; otherwise if $F_1^c \in \mathcal{U}$ take $I_2 := \overline{R_1}$. In any case I_2 is a compact interval with the property that $\{n \in \mathbb{N} : x_n \in I_2\} \in \mathcal{U}$ (for the case $I_2 := \overline{R_1}$, use the just proved property (iii).) Next we consider c_2 to be the midpoint of I_2 and proceed similarly.
In the end, we obtain a decreasing sequence of compact interval (I_n) with $\lim \ell(I_n) = 0$ where $\ell(\cdot)$ denots the length of an interval. Furthermore $\{n \in \mathbb{N} : x_n \in I_k\} \in \mathcal{U}$ for all $k \geq 2$. By the nested interval theorem, $\bigcap I_n = \{L\}$ for some $L \in \mathbb{R}$. We finally show that L is the required number (ultra-limit) with respect to (the ultra-filter) $\mathcal{U}$.
Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $\ell(I_n) < \epsilon$ for all $n \geq N$. It follows that $\{n \in \mathbb{N} : x_n \in I_N\} \subset \{n \in \mathbb{N} : |x_n - L| < \epsilon\}$ as $L \in I_N$. It follows from property (iii) that $\{n \in \mathbb{N} : |x_n - L| < \epsilon\} \in \mathcal{U}$.

2. (20 marks)

(a) Let $f : [a, b] \to \mathbb{R}$ be a function. We say $c \in [a, b]$ a jumping point of f if c is a discontinuous point and either $f(c) = \lim_{x\to c^+} f(x)$ or $f(c) = \lim_{x\to c^-} f(x)$. Now suppose f is continuous on $[a, b]$ except for finitely many jumping points. Show that f is a point-wise limit of continuous functions, that is, there exist a sequence of continuous functions $(f_n : [a, b] \to \mathbb{R})$ such that $f(x) = \lim_n f_n(x)$ for all $x \in [a, b]$.

(b) Let $g : [a, b] \to \mathbb{R}$ be a real-valued function. Suppose g is a point-wise limit of continuous functions, that is, there exists a sequence of continuous functions (g_n) on $[a, b]$ such that $g(x) = \lim_n g_n(x)$ for all $x \in [a, b]$. Show that for all $m, M \in \mathbb{R}$ with $m < M$ the set $F := \{x \in [a, b] : m < g(x) < M\}$ is a countable union of compact sets, that is, there exists a sequence of compact sets (K_n) such that $F = \bigcup_{n=1}^{\infty} K_n$

Solution.

(a) First we consider the case where f is continuous on $[a, b]$ except for 1 jump point $c \in [a, b]$. Note that $c \neq a, b$ as end-points cannot be jump points since one-sided limits exist. Now pick a sequence of decreasing positive number (r_n) with $\lim r_n = 0$ such that $I_n := [a_n, b_n] := [-r_n + c, c + r_n] \subset [a, b]$, for example we can take some tail subsequence of the sequence $(1/n)$. Now we define the sequence of continuous functions as follows:

$$g_n(x) := \begin{cases} f(x) & x \notin I_n \cap [a, b] \\ h_n(x) & x \in I_n \cap [a, b] \end{cases}$$

where $h_n : I_n \to \mathbb{R}$ are some continuous functions agreeing with f on a_n, c, b_n, for example, we can take the unique piecewise linear function connecting these three points. It is easy to see that (g_n) are continuous functions on $[a, b]$, for example by considering two-sided limits. (A little care may be needed when considering end-points of I_n). Now we are showing that $g(x) = \lim g_n(x)$ for all $x \in [a, b]$. If $x \neq c$, it is clear as g_n is eventually the same as f due to the fact that (I_n) is shrinking. When $x = c$, then by construction $g_n(x) = g_n(c) = h_n(c) = f(c)$ for all $n \in \mathbb{N}$. It is clear that $\lim g_n(c) = \lim f(c) = f(c)$. It follows that $\lim g_n(x) = f(x)$ for all $x \in [a, b]$

Next we consider the case where f is continuous except for finitely many jump points. In fact, the proof is similar to the case with a single discontinuity except that now the sequence of continuous functions are defined piecewise in more regions whose number equals to the number of discontinuity points (plus 1). Furthermore, we choose the radii of neighborhoods of the discontinuity points to be small enough such that they are disjoint. This is possible due to finiteness of the discontinuity points. We skip the detailed proof in this brief solution.

(b) First note that the set $F := \{x \in [a, b] : m < g(x) < M\} \subset [a, b]$ is bounded. It suffcies to show that F is a countable union of closed sets (as compact sets are equivalent to closed and bounded subsets on $\mathbb{R}$ by Heine-Borel Theorem). Next we claim that

- For $t \in [a, b]$, we have $m < g(t) < M$ if and only if there exists $\epsilon > 0$ and there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $m + \epsilon \leq g_n(t) \leq M - \epsilon$.

($\Rightarrow$). First, we take arbitrary $\epsilon > 0$ such that $m + \epsilon < g(t) < M - \epsilon$, for example, we can choose to take $\epsilon < \min\{g(t) - m, M - g(t)\}$. Now we have $m + \epsilon < \lim_n g_n(t) < M - \epsilon$. By limit property on strict inequality, we have $m + \epsilon < g_n(t) < M - \epsilon$ eventually, that is, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $m+\epsilon < g_n(t) < M-\epsilon$. The result follows by considering the partial inequalities, that is, $m+\epsilon \leq g_n(t) \leq M-\epsilon$.
($\Leftarrow$). Now suppose the condition holds. Then $m + \epsilon \leq g_n(t) \leq M - \epsilon$ eventually (respect to n). As it is given that $g(t) = \lim_n g_n(t)$, we have $m + \epsilon \leq g(t) \leq M - \epsilon$ as $n \to \infty$. This implies that we can have the inequality $m < m + \epsilon \leq g(t) \leq M - \epsilon < M$.

With this characterization, it follows that we have

$$\{t \in [a, b] : g(t) \in (m, M)\} = \bigcup_{\epsilon>0} \bigcup_{N\in\mathbb{N}} \bigcap_{n\geq N} g_n^{-1}([m + \epsilon, M - \epsilon])$$

by translating "there exists" to unions and "for all" to intersections. In fact by density of rational numbers, we can replace $\epsilon > 0$ by considering only the positive rational numbers, hence, we have

$$\{t \in [a, b] : g(t) \in (m, M)\} = \bigcup_{q>0, q\in\mathbb{Q}} \bigcup_{N\in\mathbb{N}} \bigcap_{n\geq N} g_n^{-1}([m + q, M - q])$$

and leave the verification for the readers by slight modifying the proof of the above claim.
After that, we show that $g_n^{-1}(M + q, M - q)$ for all $q > 0, q \in \mathbb{Q}$ and $n \in \mathbb{N}$ is a closed set. This follows from continuity of g_n's as continuous preimages of closed sets are closed (cf. taking complement in Tutorial 8, P. 1, QP1). Furthermore as arbitrary intersection of closed sets is closed (see Tutorial 8 again), it follows that the set in question is a countable union of closed sets. Precisely, we have

$$\{t \in [a, b] : g(t) \in (m, M)\} = \underbrace{\bigcup_{q>0, q\in\mathbb{Q}} \bigcup_{N\in\mathbb{N}}}_{\textit{Countable Union}} \underbrace{\bigcap_{n\geq N} g_n^{-1}([m + q, M - q])}_{\textit{Closed set}}$$