MATH 2058 - HW 4 - Solutions

1 (P.84 Q4)**.** Show that the following sequences $x = (x_n)$ are divergent.

a) $x_n := 1 - (-1)^n + 1/n$ b) $x_n := \sin(n\pi/4)$

Solution. For both sequences, it suffices to consider subsequences that converge to different limits.

a. Consider the subsequence (y_n) where $y_n := x_{2n}$. Then $y_n = x_{2n} = 1 - 1 + 1/2n = 1/2n$ for all $n \in \mathbb{N}$. It follows that $\lim y_n = \lim 1/2n = 1/2 \lim 1/n = 0$.
Next, consider the subsequence (z_n) where $z_n = x_{2n+1}$ for all $n \in \mathbb{N}$. Then $z_n = 1 - (-1) + 1/(2n+1) = 2 + 1/(2n+1)$. Then $\lim z_n = \lim 2 + 1/(2n+1) = \lim 2 + \lim 1/(2n+1) = 2 + 0 = 2$ where $\lim 1/2n + 1 = 0$ by considering it as subsequential limit of $(1/n)$. To conclude $\lim y_n = 0 \neq \lim z_n = 2$ and so (x_n) have subsequences that converge differently. By consistency of subsequential limits, (x_n) diverges.

b. Consider the subsequence (y_n) where $y_n := x_{8n}$ for all $n \in \mathbb{N}$. Then $y_n = \sin(2n\pi) = 0$ for all $n \in \mathbb{N}$. Hence $\lim y_n = 0$.
Next, consider the subsequence (z_n) where $z_n = x_{8n+1}$ for all $n \in \mathbb{N}$. Then $z_n = \sin(2n\pi + \pi/4) = \sin(\pi/4) = \sqrt{2}/2$ for all $n \in \mathbb{N}$. Hence $\lim z_n = \sqrt{2}/2$.
To conclude, $\lim y_n = 0 \neq \sqrt{2}/2 = \lim z_n$ and so (x_n) have subsequences that converge differently. By consistency of subsequential limits, (x_n) diverges.

2. Let (x_n) be an unbounded sequence. Show that there exists a subsequence (y_n) of (x_n) such that $\lim 1/y_n = 0$.

Solution. Note that (x_n) is unbounded if and only if for all $M > 0$, there exists $n \in \mathbb{N}$ such that $|x_n| \geq M$ by considering the negation of a bounded sequence. We observe that if (x_n) is unbounded then every **tail subsequence**, which is of the form $(x_n)_{n \geq k}$ for some $k \in \mathbb{N}$ is **unbounded**. This is because suppose there exists a bounded tail subsequence, say $(x_n)_{n \geq k}$. Then there exists $M > 0$ such that $|x_n| \leq M$ for all $n \geq k$. It then follows that $|x_n| \leq \max\{|x_1|, \cdots, |x_{k-1}|, M\}$ and so (x_n) is bounded, which is a contradiction.
We then proceed to construct the required subsequence. First consider $M = 1$. Then by the above characterization, there exists $N(1)$ such that $\left|x_{N(1)}\right| \geq 1$. Now note that the tail sequence $(x_n)_{n > N(1)}$ is again unbounded. Consider $M = 2$. Then there exists $N(2) \in \mathbb{N}$ such that $\left|x_{N(2)}\right| \geq 2$ **and** $N(2) > N(1)$. Inductively, for all $n \geq 3$, we consider the tail sequence $(x_n)_{n > N(n-1)}$ which is unbounded. Therefore, there exists $N(n)$ with $N(n) > N(n-1)$ such that $\left|x_{N(n)}\right| \geq n$. Therefore, define $y_n := x_{N(n)}$ for all $n \in \mathbb{N}$. Then (y_n) is a subsequence of (x_n) as the map $n \mapsto N(n)$ is strictly increasing by the construction. Moreover, we have $\left|x_{N(n)}\right| \geq n$ for all $n \in \mathbb{N}$, which implies that

$$0 \leq \frac{1}{|y_n|} = \frac{1}{\left|x_{N(n)}\right|} \leq \frac{1}{n}$$

Since, $\lim 1/n = 0$, by Squeeze Theorem, it follows that $\lim 1/|y_n| = 0$. Considering $-|y_n| \leq |y_n| \leq |y_n|$ for all $n \in \mathbb{N}$, we obtain $\lim 1/y_n = 0$ by Squeeze Theorem again.

Comment. It is not okay to simply construct a subsequence (y_n) by declaring that $y_n = x_{k(n)}$ where $\left|x_{k(n)}\right| \geq n$. We do not know if $\{k(n) : n \in \mathbb{N}\}$ has repetition and so $k : \mathbb{N} \to \mathbb{N}$ may not have a range that is infinite, let alone strictly increasing. One way to get around with the issue is to consider tail subsequences.