MATH 2058 - HW 3 - Solutions

1 (P.77 Q4)**.** Let (x_n) be a sequence with $x_1 := 1$ and $x_{n+1} := \sqrt{2 + x_n}$ for all $n \in \mathbb{N}$.

a. Show that (x_n) converges.

b. Find $\lim x_n$

Solution.

a. We first show that (x_n) is increasing. We proceed using induction. Note $x_2 = \sqrt{2+1} = \sqrt{3} \geq 1 = x_1$. Now fix $n \in \mathbb{N}$ and suppose $x_n \geq x_{n-1}$. We want to show that $x_{n+1} \geq x_n$. Note that $x_n = \sqrt{x_{n-1} + 2}$ and so $x_n^2 = x_{n-1} + 2 \leq x_n + 2$. Therefore, $x_n \leq \sqrt{x_n + 2}$. As we have $x_{n+1} = \sqrt{x_n + 2}$, we have $x_{n+1} \geq x_n$. It follows by induction that $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$ and so (x_n) is increasing.
Next we show that (x_n) is bounded above by 2 and again we proceed with induction. Note that $x_1 = 1 \leq 2$. Now fix $n \in \mathbb{N}$ and suppose $x_n \leq 2$. Then $x_{n+1} = \sqrt{2 + x_n} \leq \sqrt{2+2} = 2$. Hence by induction, $x_n \leq 2$ for all $n \in \mathbb{N}$.
Lastly, as (x_n) is bounded above increasing, by the bounded monotone convergence theorem, it follows that (x_n) converges.

b. Write $x := \lim x_n$. Note that $(y_n := x_{n+1})$ is a subsequence of (x_n) and we have $y_n = x_{n+1} = \sqrt{2 + x_n}$ for all $n \in \mathbb{N}$. Therefore we have

$$x = \lim y_n = \lim \sqrt{2 + x_n} = \sqrt{2 + x}$$

as limit is preserved by square roots, addition and subsequences. Hence we have $x^2 = 2 + x$, which implies we have $x = 2$ or $x = -1$. The latter is easily rejected as (x_n) is increasing with $x_1 = 1$. (In fact, $x = -1$ is not even a solution to $x = \sqrt{2 + x}$.) Therefore it must be the case that $\lim x_n = x = 2$

2 (P.77 Q10)**.** Establish the convergence or divergence of the sequence (y_n) where

$$y_n := \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$$

for all $n \in \mathbb{N}$

Solution. We claim that (y_n) converges.
First we observe that for all $n \in \mathbb{N}$, y_n is the sum of $(2n - (n+1) + 1 = n)$ unit fractions. Therefore, we have

$$y_n = \underbrace{\frac{1}{n+1} + \cdots + \frac{1}{2n}}_{\text{n terms}} \leq \underbrace{\frac{1}{n+1} + \cdots + \frac{1}{n+1}}_{\text{n terms}} = \frac{n}{n+1} \leq 1$$

for all $n \in \mathbb{N}$. Hence, (y_n) is bounded above.
Furthermore, for all $n \in \mathbb{N}$, we have

$$\begin{aligned} y_{n+1} - y_n &= (\frac{1}{n+2} + \cdots + \frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{2n+2}) - (\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}) \\ &= \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} = \frac{1}{2n+1} - \frac{1}{2n+2} = \frac{1}{(2n+1)(2n+2)} \geq 0 \end{aligned}$$

It follows that $y_{n+1} \geq y_n$ for all $n \in \mathbb{N}$ and so (y_n) is increasing.
To conclude, (y_n) is bounded above increasing and so by the bounded monotone convergence theorem, (y_n) converges.

Remark. To establish the convergence or divergence of complicated sequences like (y_n) in Q2, it is a good practice to first familiarize yourself with the sequence by writing its first few terms out.