MATH 2058 - HW 2 - Solutions

1 (P.61-62 Q5cd)**.** Establish the following limits using the definition of limit.

a) $\lim \frac{3n+1}{2n+5} = \frac{3}{2}$ b) $\lim \frac{n^2-1}{2n^2+3} = \frac{1}{2}$

Solution.

a. Let $\epsilon > 0$. By Archimedean Property, there exists $N \in \mathbb{N}$ such that $13/N < \epsilon$. Now suppose $n \geq N$. Then we have

$$\left|\frac{3n+1}{2n+5} - \frac{3}{2}\right| = \left|\frac{(6n+2)-(6n+15)}{2(2n+5)}\right| = \left|\frac{13}{2(2n+5)}\right| = \frac{13}{4n+10} \leq \frac{13}{n} \leq \frac{13}{N} \leq \epsilon$$

We have established the sequential limit by definition.

b. Let $\epsilon > 0$. By Archimedean Property, there exists $N \in \mathbb{N}$ such that $5/N < \epsilon$. Now suppose $n \geq N$. Then we have

$$\left|\frac{n^2-1}{2n+3} - \frac{1}{2}\right| = \left|\frac{(2n^2-2)-(2n^2+3)}{2(2n^2+3)}\right| = \frac{5}{2(2n^2+3)} = \underbrace{\frac{5}{4n^2+6}}_{:=(*)} \leq \frac{5}{n^2} \leq \frac{5}{n} \leq \frac{5}{N}\epsilon$$

We have established the sequential limit by definition.

2 (P.61 - 62 Q9)**.** Let (x_n) be a sequence such that $x_n \geq 0$ for all $n \in \mathbb{N}$. Suppose $\lim x_n = 0$. Show that $\lim \sqrt{x_n} = 0$

Solution. Let $\epsilon > 0$. Since $\lim x_n = 0$, there exists $N \in \mathbb{N}$ such that $|x_n| \leq \epsilon^2$ for all $n \geq N$. Note that for all $x, y \in \mathbb{R}$ with $x, y \geq 0$, we have $x \leq y$ if and only if $x^2 \leq y^2$ (why?). Now suppose $n \geq N$. By the previous remark, we then have $\sqrt{x_n} \leq \epsilon$ as $\sqrt{x_n}^2 \leq \epsilon^2$. Hence, $\left|\sqrt{x_n} - 0\right| = \sqrt{x_n} \leq \epsilon$ for all $n \geq N$. It follows from definition that $\lim \sqrt{x_n} = 0$

Remark. Sometimes, one may want to find a suitable N using the Archimedean Property before making simplification to the distance term $|x_n - x|$. To do this, we may solve the inequality $|x_n - x| < \epsilon$ using n as the variable. For example, in Q1b, we can find an N directly at the expression $(*)$.
To do this, we have to solve the inequality

$$\frac{5}{4n^2+6} < \epsilon$$

with respect to n, which is equivalent to

$$\frac{5-6\epsilon}{4\epsilon} < n^2$$

which is in turn equivalent to

$$0 \leq \sqrt{\frac{5-6\epsilon}{4\epsilon}} < n$$

if $0 \leq 5 - 6\epsilon \iff \epsilon \leq 5/6$. Therefore, to proceed with the Archimedean Property, we have to first add the assumption $0 < \epsilon \leq 5/6$ so that we can find an $N \in \mathbb{N}$ such that $\sqrt{\frac{5-6\epsilon}{4\epsilon}} < N$. We can then claim that $\frac{5}{4N^2+6} < \epsilon$. Furthermore, since the expression $\frac{5}{4n^2+6}$ is decreasing in terms of n (which is not hard to see for this question), we can then claim that as $n \geq N$, we have

$$\frac{5}{4n^2+6} \leq \frac{5}{4N^2+6} < \epsilon$$

As you might see, things would become complicated if we do not want to simplify our inequalities first before using the Archimedean Property. I recommend always simplifying your inequality first.