## THE CHINESE UNIVERSITY OF HONG KONG Department of Mathematics MATH 2070A Algebraic Structures 2018-19 Homework 9 Due Date: 22nd November 2018

## **Compulsory Part**

- 1. Let R be a commutative ring. Let u be a unit in R, show that R/(u) is isomorphic to the zero ring  $\{0\}$ .
- 2. Let a, b be integers. Show that  $\mathbb{Z}[i]/(a+bi) \cong \mathbb{Z}[i]/(a-bi)$  by performing the following steps:
  - (a) Define  $\phi : \mathbb{Z}[i] \to \mathbb{Z}[i]/(a-bi)$  as follows:

 $\phi(c+di) = \overline{c-di} := c - di + (a - bi), \quad c, d \in \mathbb{Z}.$ 

Show that  $\phi$  is a ring homomorphism.

- (b) Show that  $\phi$  is surjective.
- (c) Show that the kernel of  $\phi$  is (a + bi).
- (d) Apply the First Isomorphism Theorem for rings.
- 3. Is  $\mathbb{Q}[x]/(x-1)$  isomorphic to  $\mathbb{Q}[x]/(x+1)$ ? Justify your answer.

## **Optional Part**

- 1. For any natural number m > 1, show that there cannot be a homomorphism from  $\mathbb{Q}$  to  $\mathbb{Z}_m$ .
- 2. (a) How many elements are there in  $\mathbb{Z}_{12}/(3)$ ?
  - (b) How many elements are there in  $\mathbb{Z}_{12}/(5)$ ?
  - (c) How many equivalence classes are there in  $\mathbb{Z}_2[x]$  modulo the ideal generated by  $x^3 + 1$ ? Give a representative in  $\mathbb{Z}_2[x]$  for each of these equivalence classes.
- 3. In class, we showed that Z[i]/(1 + 3i) ≅ Z/10Z, where 10 happens to be equal to (1 + 3i)(1 3i) = 1<sup>2</sup> + 3<sup>2</sup>. Is Z[i]/(a + bi) always isomorphic to Z/(a<sup>2</sup> + b<sup>2</sup>), for all a, b ∈ Z? For example, is Z[i]/(2 + 2i) isomorphic to Z/8Z?

*Hint*: If  $\mathbb{Z}[i]/(2+2i)$  is isomorphic to  $\mathbb{Z}/8\mathbb{Z}$ , then it is isomorphic to  $\mathbb{Z}_8 = \{0, 1, 2, ..., 7\}$ . Any isomorphism  $\phi$  from  $\mathbb{Z}/(2+2i)$  to  $\mathbb{Z}_8$  must send 1 to 1, 0 to 0, and  $\overline{i} = i + (2+2i)$  to some  $a \in \mathbb{Z}_8$ . What properties must this a satisfy? Does there exist an  $a \in \mathbb{Z}_8$  which satisfies all these properties?

Let R = C[-1,1], the ring of continuous real-valued functions on [-1,1], equipped with the usual operations of addition and multiplication for real-valued functions. Let I = {f ∈ R : f(0) = 0}.

- (a) Show that I is an ideal in R.
- (b) Show that:

$$R/I \cong \mathbb{R}.$$

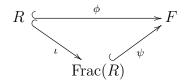
- 5. Are the rings  $\mathbb{Z}_2[x]/(x^2+1)$  and  $\mathbb{Z}_2[x]/(x^3+1)$  isomorphic? Justify your answer.
- 6. Are the rings  $\mathbb{R}[x]/(x^2)$  and  $\mathbb{R}[x]/(x^2-2x+1)$  isomorphic? Justify your answer.
- 7. Are the rings  $\mathbb{Q}[x]/(x^2)$  and  $\mathbb{Q}[x]/(x^2-1)$  isomorphic? Justify your answer.
- 8. Let R be an integral domain. Let  $\iota : R \hookrightarrow Frac(R)$  be the injective (i.e. one-to-one) homomorphism defined by:

$$\iota(r) = [(r,1)], \quad \forall r \in R.$$

Suppose there exists a field F, along with an injective homomorphism  $\phi : R \longrightarrow F$ . Show that there exists an injective homomorphism:

$$\psi: \operatorname{Frac}(R) \hookrightarrow F$$

such that  $\psi \circ \iota = \phi$ . (Terminology: In this case, we say that the diagram below is a *commutative diagram*, or that the diagram *commutes*.)



**Remark:** This result essentially says that the field of fractions of an integral domain R is the "smallest" field containing R as a subring.