THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2070A Algebraic Structures 2018-19
Homework 9
Due Date: 22nd November 2018

Compulsory Part

1. Let R be a commutative ring. Let u be a unit in R, show that $R/(u)$ is isomorphic to the zero ring $\{0\}$.

2. Let a, b be integers. Show that $\mathbb{Z}[i]/(a+bi) \cong \mathbb{Z}[i]/(a-bi)$ by performing the following steps:

   (a) Define $\phi : \mathbb{Z}[i] \to \mathbb{Z}[i]/(a-bi)$ as follows:
   $$\phi(c+di) = \overline{c-di} := c - di + (a-bi), \quad c, d \in \mathbb{Z}.$$
   Show that ϕ is a ring homomorphism.

   (b) Show that ϕ is surjective.

   (c) Show that the kernel of ϕ is $(a+bi)$.

   (d) Apply the First Isomorphism Theorem for rings.

3. Is $\mathbb{Q}[x]/(x-1)$ isomorphic to $\mathbb{Q}[x]/(x+1)$? Justify your answer.

Optional Part

1. For any natural number $m > 1$, show that there cannot be a homomorphism from $\mathbb{Q}$ to $\mathbb{Z}_m$.

2. (a) How many elements are there in $\mathbb{Z}_{12}/(3)$?

   (b) How many elements are there in $\mathbb{Z}_{12}/(5)$?

   (c) How many equivalence classes are there in $\mathbb{Z}_2[x]$ modulo the ideal generated by x^3+1? Give a representative in $\mathbb{Z}_2[x]$ for each of these equivalence classes.

3. In class, we showed that $\mathbb{Z}[i]/(1+3i) \cong \mathbb{Z}/10\mathbb{Z}$, where 10 happens to be equal to $(1+3i)(1-3i) = 1^2+3^2$. Is $\mathbb{Z}[i]/(a+bi)$ always isomorphic to $\mathbb{Z}/(a^2+b^2)$, for all $a, b \in \mathbb{Z}$?

   For example, is $\mathbb{Z}[i]/(2+2i)$ isomorphic to $\mathbb{Z}/8\mathbb{Z}$?

   Hint: If $\mathbb{Z}[i]/(2+2i)$ is isomorphic to $\mathbb{Z}/8\mathbb{Z}$, then it is isomorphic to $\mathbb{Z}_8 = \{0, 1, 2, \ldots, 7\}$. Any isomorphism ϕ from $\mathbb{Z}/(2+2i)$ to $\mathbb{Z}_8$ must send 1 to 1, 0 to 0, and $\bar{i} = i + (2+2i)$ to some $a \in \mathbb{Z}_8$. What properties must this a satisfy? Does there exist an $a \in \mathbb{Z}_8$ which satisfies all these properties?

4. Let $R = C[-1, 1]$, the ring of continuous real-valued functions on $[-1, 1]$, equipped with the usual operations of addition and multiplication for real-valued functions. Let $I = \{f \in R : f(0) = 0\}$.

(a) Show that I is an ideal in R.

(b) Show that:

$$R/I \cong \mathbb{R}.$$

5. Are the rings $\mathbb{Z}_2[x]/(x^2+1)$ and $\mathbb{Z}_2[x]/(x^3+1)$ isomorphic? Justify your answer.

6. Are the rings $\mathbb{R}[x]/(x^2)$ and $\mathbb{R}[x]/(x^2-2x+1)$ isomorphic? Justify your answer.

7. Are the rings $\mathbb{Q}[x]/(x^2)$ and $\mathbb{Q}[x]/(x^2-1)$ isomorphic? Justify your answer.

8. Let R be an integral domain. Let $\iota : R \hookrightarrow \mathrm{Frac}(R)$ be the injective (i.e. one-to-one) homomorphism defined by:

$$\iota(r) = [(r,1)], \quad \forall\, r \in R.$$

Suppose there exists a field F, along with an injective homomorphism $\phi : R \hookrightarrow F$. Show that there exists an injective homomorphism:

$$\psi : \mathrm{Frac}(R) \hookrightarrow F$$

such that $\psi \circ \iota = \phi$. (Terminology: In this case, we say that the diagram below is a *commutative diagram*, or that the diagram *commutes*.)

$$\begin{tikzcd} R \arrow[rr, hook, "\phi"] \arrow[dr, hook, "\iota"'] & & F \\ & \mathrm{Frac}(R) \arrow[ur, hook, "\psi"'] & \end{tikzcd}$$

Remark: This result essentially says that the field of fractions of an integral domain R is the "smallest" field containing R as a subring.