THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2070A Algebraic Structures 2018-19
Homework 6
Due Date: 25th October 2018

Compulsory Part

1. Prove the following identities in an arbitrary ring R:
   (a) $(-a)(-b) = ab$ for any $a, b \in R$.
   (b) $(-a)b = a(-b) = -(ab)$ for any $a, b \in R$.

2. Show that $a^2 - b^2 = (a+b)(a-b)$ for all a, b in a ring R if and only if R is commutative.

3. A ring R such that $a^2 = a$ for any $a \in R$ is called a **Boolean ring**. Show that every Boolean ring is commutative.

Optional Part

1. Let R be a commutative ring. Define the circle binary operation $\circ$ on R as follows:
$$a \circ b = a + b - ab, \quad a, b \in R.$$
Show that the circle operation is associative, and that $0 \circ a = a$ for all $a \in R$. (Here, 0 denotes the additive identity element of R.)

2. Let R be a commutative ring. Show that the **binomial theorem** holds, i.e.
$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$
for any $a, b \in R$ and for any positive integer n.

3. Let R be the set of all real-valued functions f on $\mathbb{R}$ such that $f(0) = 0$. Let $+$ and $\cdot$ be the usual addition and multiplication operations for functions.
   (a) Show that $f + g \in R$ for all $f, g \in R$.
   (b) Show that $f \cdot g \in R$ for all $f, g \in R$.
   (c) With respect to $+$, what is the additive identity element of R, if it exists?
   (d) With respect to $\cdot$, what is the multiplicative identity element of R, if it exists?

4. Let X be a set, and R is the set of subsets of X. In each of the following cases, decide whether the given operations in R form a ring:
   (a) For $A, B \in R$, we define $A + B := A \cup B$ and $A \cdot B := A \cap B$.
   (b) For $A, B \in R$, we define $A + B := (A \cup B) \setminus (A \cap B)$ and $A \cdot B := A \cap B$.