THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2070A Algebraic Structures 2018-19
Homework 2
Due Date: 20th September 2018

Compulsory Part

1. Let G be a group. Show that for all $a, b \in G$ such that $|ab|$ is finite, we have $|ab| = |ba|$.

2. Let $m \in \mathbb{N}$. Let $\xi_m = e^{2\pi i/m} \in \mathbb{C}$. Consider

$$U_m = \{1, \xi_m, \xi_m^2, \dots, \xi_m^{m-1}\},$$

equipped with multiplication of complex numbers. Show that

$$|\xi_m^j| = \frac{m}{\gcd(m, j)}$$

for $j = 1, 2, \dots, m-1$.

3. Consider the permutations

$$\sigma = (2147)(5782),\ \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 6 & 2 & 5 & 8 & 3 & 7 & 1 \end{pmatrix} \in S_8.$$

   (a) Express σ and τ as
      i. a product of transpositions, and
      ii. a product of disjoint cycles.

   (b) Compute $|\sigma|$, $|\tau|$ and $|\sigma\tau|$. (*Hint:* You may use the result in Q3(c) in the Optional Part.)

4. How many elements are of order 2 in D_8?

Optional Part

1. Let a, b be elements of a group G. Suppose a has order 5 and $a^3b = ba^3$. Prove that $ab = ba$.

2. By definition, the **orthogonal group** $\mathrm{O}(2, \mathbb{R})$ consists of real 2×2 matrices A which satisfy the condition

$$A^tA = AA^t = I,$$

where A^t denotes the transpose of A, and I is the 2×2 identity matrix.

   (a) Show that $\mathrm{O}(2, \mathbb{R})$ is a group under matrix multiplication.

   (b) Find an element of order 2 in $\mathrm{O}(2, \mathbb{R})$.

   (c) Find an element of order 3 in $\mathrm{O}(2, \mathbb{R})$.

3. (a) Show that a k-cycle in S_n has order k

   (b) Let $\mu_1, \mu_2 \in S_n$ be two disjoint cycles. Using the fact that disjoint cycles commute, show that
$$|\mu_1\mu_2| = \text{lcm}(|\mu_1|, |\mu_2|).$$

   (c) By induction, show that if $\mu_1, \mu_2, \ldots, \mu_r \in S_n$ are disjoint cycles, then
$$|\mu_1\mu_2\cdots\mu_r| = \text{lcm}(k_1, k_2, \ldots, k_r),$$
where $k_i = |\mu_i|$ for $i = 1, 2, \ldots, r$.

4. Find the order of the following elements in S_7:

   (a) (1325).

   (b) $(1325)(47)$.

   (c) $(1325)(647)$.

   (d) $(35)(46)(37)(32)$.

5. (a) How many elements are of order 3 in S_5?

   (b) How many elements are of order 4 in S_6?

6. Consider the dihedral group D_n, where n is an integer greater than 2. For $j = 0, 1, 2, \ldots, n-1$, let r_j denote the anticlockwise rotation about the origin by $2\pi j/n$.

   (a) Show that for any $k \in \mathbb{N}$, rotation $r \in D_n$ and reflection $s \in D_n$, we have
$$sr^k s = (srs)^k.$$

   (b) Show that for any reflection $s \in D_n$ and any rotation $r \in D_n$, we have
$$srs = r^{-1}.$$

   (*Hint*: First prove the identities for one particular reflection s, then use the fact that any other reflection s' is equal to sr' for some rotation $r' \in D_n$.)

7. Let G be a group in which every element has order at most 2. Show that G is abelian.

8. Let G be a finite group with an even number of elements. Show that there must be an order 2 element $a \in G$.