THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2070A Algebraic Structures 2018-19
Homework 11
Due Date: 6th December 2018

Compulsory Part

1. Determine if the following rings are fields. Justify your answers.

   (a) $\mathbb{Q}[x]/(x^{17} + 5x^2 - 10x + 45)$

   (b) $\mathbb{Z}[x]/(x^6 - 210x - 616)$. (Note: It is $\mathbb{Z}[x]$ instead of $\mathbb{Q}[x]$.)

   (c) $\mathbb{Q}[x]/(4x^3 - 6x - 1)$

   (d) $\mathbb{R}[x]/(x^{17} + 5x^2 - 10x + 45)$

2. Let p be a prime.

   (a) Show that for all $k \in \{1, 2, \ldots, p-1\}$, the prime p divides $\binom{p}{k}$.

   (b) Let p be a prime, r an element in $\mathbb{F}_p$. Show that $(x+r)^p = x^p + r^p$ in $\mathbb{F}_p[x]$.

   (c) The p**-th cyclotomic polynomial** is by definition:

   $$\Phi_p = x^{p-1} + x^{p-2} + \cdots + x + 1.$$

   Show that Φ_p is irreducible in $\mathbb{Q}[x]$.
   (Hint: First show that:

   $$\Phi_p \circ (x+1) := (x+1)^{p-1} + (x+1)^{p-2} + \cdots + (x+1) + 1$$

   is irreducible in $\mathbb{Q}[x]$.)

Optional Part

1. Let F be a field, p a polynomial in $F[x]$. Then a theorem in our lecture notes says that the quotient ring $F[x]/(p)$ is a field if and only if p is irreducible in $F[x]$.

   Determine if each of the following rings is a field:

   (a) $\mathbb{Q}[x]/(x^3 - 1)$

   (b) $\mathbb{Q}[x]/(7x^{59} + 24x^9 + 6x + 156)$

   (c) $\mathbb{Q}[x]/(x^3 + x + 1)$

   (d) $\mathbb{Z}[x]/(x^3 + x + 1)$

   (e) $\mathbb{Q}/(17)$

   (f) $\mathbb{Z}/(17)$

   (g) $\mathbb{Z}[x]/(2, x)$

   (h) $\mathbb{Q}[x]/(x^2 - 3)$

(i) $\mathbb{R}[x]/(x^2 - 3)$

(j) $\mathbb{R}[x]/(x^2 + 3)$

(k) $\mathbb{F}_5[x]/(x^2 + 1)$

(l) $\mathbb{R}[x]/(x^{17} + x^5 + 8x^2 - x + 1)$

2. (a) Let a be a rational number. Show that the quotient ring $\mathbb{Q}[x]/(x - a)$ is isomorphic to $\mathbb{Q}$ by explicitly defining an isomorphism:

$$\psi : \mathbb{Q} \longrightarrow \mathbb{Q}[x]/(x - a).$$

(b) Show that $\mathbb{R}[x]/(x^2 + 1)$ is isomorphic to $\mathbb{R}[x]/(x^2 + 2)$ by explicitly defining an isomorphism.