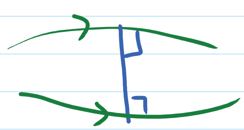
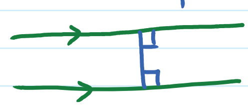
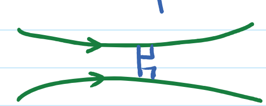
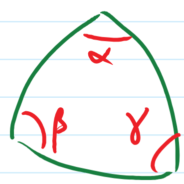
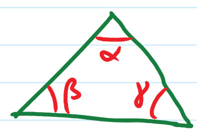
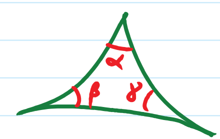
§ Comparing elliptic geometry, Euclidean geometry and hyperbolic geometry

Geometry	Elliptic	Euclidean	Hyperbolic
underlying space	real projective plane $\mathbb{RP}^2 = S^2/\sim$ where $\sim$ means antipodal pts are identified $(\mathbb{D} \subset \mathbb{RP}^2)$	$\mathbb{C} = \mathbb{R}^2$ $\cup$ $\mathbb{D}$	$\mathbb{D}$ (or $\mathbb{U}$)
group of transform^ns	$S = \{T \in M$ of the form $T(z) = \frac{az+b}{-\bar{b}z+\bar{a}}$ where $\|a\|^2+\|b\|^2=1\}$ (or $T(z) = e^{i\theta}\frac{z-z_0}{1+\bar{z}_0 z}$ $\theta \in \mathbb{R},\ z_0 \in \mathbb{C}$)	$E = \{T \in M$ of the form $T(z) = e^{i\theta}z+b,$ $\theta \in \mathbb{R}, b \in \mathbb{C}\}$	$H = \{T \in M$ of the form $T(z) = \frac{az+b}{\bar{b}z+\bar{a}}$ where $\|a\|^2-\|b\|^2=1\}$ (or $T(z) = e^{i\theta}\frac{z-z_0}{1-\bar{z}_0 z}$ $\theta \in \mathbb{R},\ z_0 \in \mathbb{D}$)
geodesics (or straight lines)	(part of) great circles (also called elliptic straight lines)	Euclidean straight lines	hyperbolic straight lines
Euclid's postulates	P1 – P4 ✓ P5 ✗ $\nexists$ parallel lines 	P1 – P4 ✓ P5 ✓ $\exists!$ line parallel to a given line thru a pt 	P1 – P4 ✓ P5 ✗ $\exists$ 2 lines parallel to a given line thru a pt 
length of a smooth curve	ℓ ...	ℓ ...	$\ell(\gamma) = \int^b \frac{2\|z'(t)\|dt}{\ldots}$

length of a smooth curve $\gamma: z(t)$, $a \le t \le b$	$\ell(\gamma) = \int_a^b \frac{2\lvert z'(t)\rvert}{1+\lvert z(t)\rvert^2} dt$	$\ell(\gamma) = \int_a^b \lvert z'(t)\rvert dt$	$\ell(\gamma) = \int_a^b \frac{2\lvert z'(t)\rvert dt}{1-\lvert z(t)\rvert^2}$
area of a region R	$A = \iint_R \frac{4r\,dr\,d\theta}{(1+r^2)^2} = \iint_R \frac{4\,dx\,dy}{(1+x^2+y^2)^2}$	$A = \iint_R r\,dr\,d\theta = \iint_R dx\,dy$	$A = \iint_R \frac{4r\,dr\,d\theta}{(1-r^2)^2} = \iint_R \frac{4\,dx\,dy}{(1-x^2-y^2)^2}$
area of triangle Δ with angles α, β, γ	$A(\Delta) = (\alpha+\beta+\gamma) - \pi$ (angular excess)	$A(\Delta)$ NOT related to angle sum	$A(\Delta) = \pi - (\alpha+\beta+\gamma)$ (angular defect)
angle sum of a triangle Δ	 $\alpha+\beta+\gamma > \pi$	 $\alpha+\beta+\gamma = \pi$	 $\alpha+\beta+\gamma < \pi$
curvature $\equiv$ const.	1	0	-1

§ An interpolation between these 3 geometries

i.e. a family of geometries connecting the elliptic, Euclidean (parabolic), and hyperbolic geometries.

Let $k \in [-1, 1]$.

- (underlying space) Let $\mathbb{C}_k := \hat{\mathbb{C}}/\sim$
  where $z_1 \sim z_2$ iff $k \cdot z_1 \bar{z}_2 + 1 = 0$
- (group of [illegible] T is of the form

transformations) $G_k := \{ T \in M :$ $T(z) = \dfrac{az+b}{-k\bar{b}z+\bar{a}}$ where $|a|^2 + k|b|^2 = 1 \}$

e.g. $G_1 = S$, $G_{-1} = H$, $G_0 = E$

$\Rightarrow$ $(\mathbb{C}_k, G_k)$ defines a geometry, whose curvature is given by k.

- In $(\mathbb{C}_k, G_k)$, a straight line/geodesic is a cline C s.t. $z \in C \Rightarrow -\frac{1}{k\bar{z}} \in C$.

  e.g. when $k=0$, this means $\infty \in C \Rightarrow C$ is a Euclidean straight line.

- For a smooth curve $\gamma \subset \mathbb{C}_k$, its length is given by

$$\ell(\gamma) = \int_a^b \frac{2|z'(t)|}{1 + k|z(t)|^2}\, dt .$$

  For a region $R \subset \mathbb{C}_k$, its area is given by

$$A = \iint_R \frac{4\, r\, dr\, d\theta}{(1+kr^2)^2} = \iint_R \frac{4\, dx\, dy}{(1+k(x^2+y^2))^2}$$

- P1 – P4 hold for $(\mathbb{C}_k, G_k)$.

- relation between angle sum and area of a triangle:

$$k \cdot A(\Delta) = (\alpha + \beta + \gamma) - \pi$$

$\rightsquigarrow$ $(\mathbb{C}_k, G_k)_{k \in [-1,1]}$ is a family of geometries s.t.

$$\begin{cases} (\mathbb{C}_1, G_1) = (\mathbb{RP}^2, S) \\ (\mathbb{C}_0, G_0) = (\hat{\mathbb{C}}, E) \\ (\mathbb{C}_{-1}, G_{-1}) = (\mathbb{D}, H) \end{cases}$$

Def A geometry (S, G) is called

- homogeneous if for any $a, b \in S$, there exists
  $T \in G$ s.t. $T(a) = b$.

  ($\Leftrightarrow$ the group G is acting on S transitively)

- metric if $\exists\, d : S \times S \to \mathbb{R}$
  s.t. (1) $d(a,b) \geq 0 \;\forall a, b \in S$ and $d(a,b) = 0$ iff $a = b$ (positive definite)
  (2) $d(a,b) = d(b,a) \;\forall a, b \in S$ (symmetric)
  (3) $d(a,c) \leq d(a,b) + d(b,c) \;\forall a, b, c \in S$ (the Δ inequality)

  (i.e. (S, d) is a metric space)

- isotropic if G contains all rotations about every point in S.

Their meanings:

homogeneous $\Rightarrow$ the geometry looks the same at every pt

metric $\Rightarrow$ (S, d) is a metric space

isotropic $\Rightarrow$ the geometry (S, G) looks the same in every direction

Thm The only plane geometries which are metric, homogeneous and isotropic are elliptic geometry, parabolic (i.e. Euclidean) geometry and hyperbolic geometry.

§ Riemannian geometry

Let S be a surface. (or a higher-dimensional space/manifold e.g. $\mathbb{R}^n$)

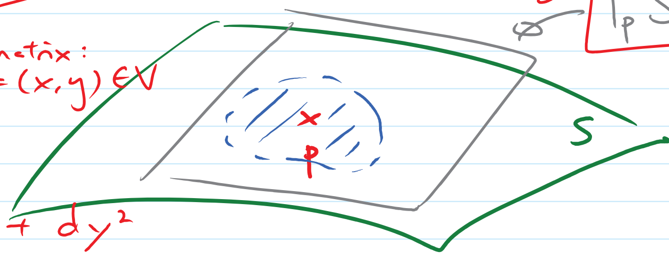
Riemann's idea: A geometry on S can be described by specifying an inner product on each tangent space to S.

an inner product on $V \cong \mathbb{R}^2$ is described by a matrix: $\vec{v} = (x,y) \in V$

$$|\vec{v}|^2 = (x,y)\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = ax^2 + 2bxy + dy^2$$

$\boxed{T_pS \cong \mathbb{R}^2}$

vector space$/\mathbb{R}$



Def A Riemannian metric on S is an inner product g_p on the tangent space T_pS at each pt $p \in S$ s.t. g_p varies smoothly as p varies

In more technical terms,

a Riemannian metric is a smooth section of the bundle $\mathrm{Sym}^2 T^*S$, where T^*S is the cotangent bundle of S, which is +ve definite.

More concretely, a Riemannian metric is a map

$$\begin{aligned} S &\longrightarrow \mathrm{Sym}^2 T^*S \\ p &\longmapsto T_pS \times T_pS \longrightarrow \mathbb{R} \\ &\qquad\qquad (u,v) \longmapsto g_p(u,v) \end{aligned}$$

In practice, by choosing a local chart and a local coordinate system, we can write the Riemannian metric as

$$g = (g_{ij}) = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where a, b, c, d are smooth functions and $b = c$.

So we can write the Riemannian metric as

$$ds^2 = (dx \ dy)\begin{pmatrix} a & b \\ b & d \end{pmatrix}\begin{pmatrix} dx \\ dy \end{pmatrix}$$

$$= a\,dx^2 + 2b\,dx\,dy + d\,dy^2$$

e.g.

	Elliptic	Euclidean (parabolic)	Hyperbolic
Riemannian metric	$ds^2 = \frac{4(dx^2+dy^2)}{(1+x^2+y^2)^2}$	$ds^2 = dx^2 + dy^2$	$ds^2 = \frac{4(dx^2+dy^2)}{(1-x^2-y^2)^2}$
	$g = \begin{pmatrix} \frac{4}{(1+x^2+y^2)^2} & 0 \\ 0 & \frac{4}{(1+x^2+y^2)^2} \end{pmatrix}$	$g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$g = \begin{pmatrix} \frac{4}{(1-x^2-y^2)^2} & 0 \\ 0 & \frac{4}{(1-x^2-y^2)^2} \end{pmatrix}$

e.g. In the upper half-plane $\mathbb{U} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, we have

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

$$\left(\text{or} \quad g = \begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix} \right)$$

Thm Every smooth surface (or manifold) admits a Riemannian metric

Given a smooth curve $\gamma \subset S$ parametrized by $z(t)$, $a \le t \le b$, its **length** is defined as

$$\ell(\gamma) := \int_a^b \|z'(t)\|\,dt$$

where $\|z'(t)\|$ is the norm of the tangent vector $z'(t) \in T_{z(t)}S$, measured by the Riemannian metric g.

Namely, in a local chart, we have

$$\|z'(t)\| = \sqrt{g(z'(t), z'(t))}$$

$$= \sqrt{\begin{pmatrix} x'(t) & y'(t) \end{pmatrix} \begin{pmatrix} a & b \\ b & d \end{pmatrix} \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}} \quad \text{if } z(t) = (x(t), y(t))$$

$$= \sqrt{a\, x'(t)^2 + 2b\, x'(t) y'(t) + d \cdot y'(t)^2}$$

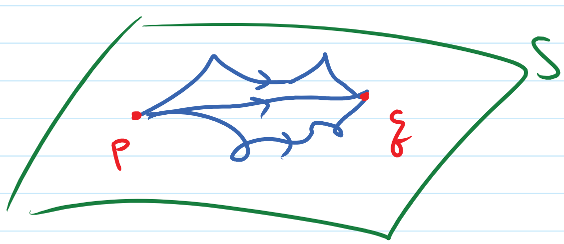
$$\Rightarrow \ell(\gamma) = \int_a^b \sqrt{a x'(t)^2 + 2b x'(t) y'(t) + d y'(t)^2}\, dt$$

e.g. for $ds^2 = \dfrac{dx^2 + dy^2}{y^2}$ on $\mathbb{U}$,

$$\ell(\gamma) = \int_a^b \sqrt{\frac{x'(t)^2}{y(t)^2} + \frac{y'(t)^2}{y(t)^2}}\, dt = \int_a^b \frac{|z'(t)|\, dt}{y(t)}$$

For any Riemannian metric g on S, we define, for any two pts $p, q \in S$,

$$d_g(p,q) := \inf \left\{ \ell(\gamma) : \begin{array}{l} \gamma \text{ is a piecewise smooth} \\ \text{curve from } p \text{ to } q \end{array} \right\}$$



Thm The function $d : S \times S \to \mathbb{R}$ is a metric on S

i.e. d is
- +ve definite,
- symmetric, and
- satisfies the Δ inequality

· satisfies the Δ inequality

So (S, d) is a metric space.