MMAT 5120 Topics in Geometry

Lecture 7

§ Hyperbolic geometry

This is the non-Euclidean geometry discovered by Gauss, Bolyai and Lobatchevsky (independently).

We will discuss two models of plane hyperbolic geometry:

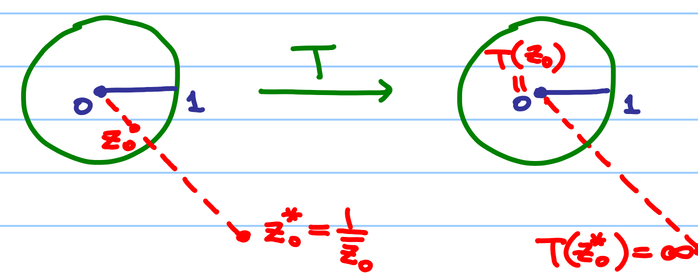
- the disk model on the open unit disk
  $$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$$
- the upper half-plane model on the upper half-plane
  $$\mathbb{U} := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$$

The disk model

Def The group H of transformations consists of all Möbius transformations that map $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ onto itself.

It is not hard to see that H is indeed a transformation group (Exercise).

To have an explicit description, let $T \in M$ which maps $\mathbb{D}$ onto itself. Then $|z| < 1 \Leftrightarrow |T(z)| < 1$, which implies that $T(\partial\mathbb{D}) = \partial\mathbb{D}$, where $\partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle.



Also, $\exists\, z_0 \in \mathbb{D}$ s.t. $T(z_0) = 0$.

$$\Rightarrow T\left(\frac{1}{\bar{z}_0}\right) = T(z_0^*) = (T(z_0))^* = 0^* = \infty$$

where z, z^* are symmetric w.r.t. $\partial\mathbb{D}$.

Hence, we have

$$T(z) = \alpha \cdot \frac{z - z_0}{z - \frac{1}{\bar{z}_0}} \qquad \text{for some } \alpha \in \mathbb{C}^*$$

$$= -\alpha \bar{z}_0 \frac{z - z_0}{1 - \bar{z}_0 z}$$

$$= \lambda \frac{z - z_0}{1 - \bar{z}_0 z}$$

where $\lambda = -\alpha \bar{z}_0 \in \mathbb{C}^*$.

When $|z| = 1$, we have $|T(z)| = 1$ and

$$|z - z_0| = |\bar{z} - \bar{z}_0| = |\bar{z} - \bar{z}_0 z \bar{z}| = |\bar{z}|\,|1 - \bar{z}_0 z| = |1 - \bar{z}_0 z|$$

$$\Rightarrow \quad 1 = |T(z)| = |\lambda| \cdot \frac{|z - z_0|}{|1 - \bar{z}_0 z|} = |\lambda|$$

$$\Rightarrow \quad \lambda = e^{i\theta} \quad \text{for some } \theta \in \mathbb{R}.$$

Therefore, we conclude that

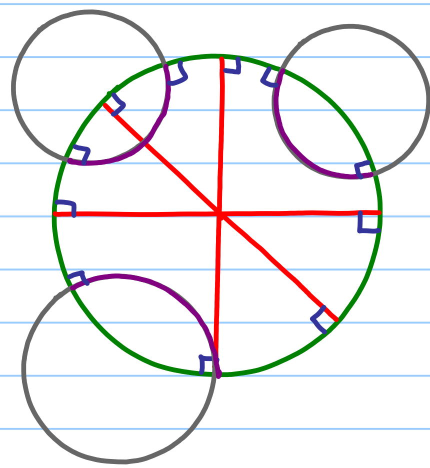
$$H = \left\{ T \in M : \begin{array}{l} T \text{ is of the form } T(z) = e^{i\theta} \dfrac{z - z_0}{1 - \bar{z}_0 z} \\ \text{for some } z_0 \in \mathbb{D} \text{ and } \theta \in \mathbb{R} \end{array} \right\}$$

The pair $(\mathbb{D}, H)$ models hyperbolic geometry.
The set $\mathbb{D}$ will be called the hyperbolic plane, and the group H is called the hyperbolic group.

Rmk H is a so-called subgroup of the Möbius group M and $\mathbb{D} \subset \hat{\mathbb{C}}$
$\Rightarrow$ $(\mathbb{D}, H)$ is a sub-geometry of $(\hat{\mathbb{C}}, M)$. In particular, every true statement in Möbius geometry remains true in hyperbolic geometry.

Hyperbolic straight lines

Def A (hyperbolic) straight line is (the part inside $\mathbb{D}$ of) a Euclidean circle or Euclidean straight line in $\mathbb{C}$ that intersects the unit circle $\partial\mathbb{D}$ at right angles.



Thm In hyperbolic geometry,

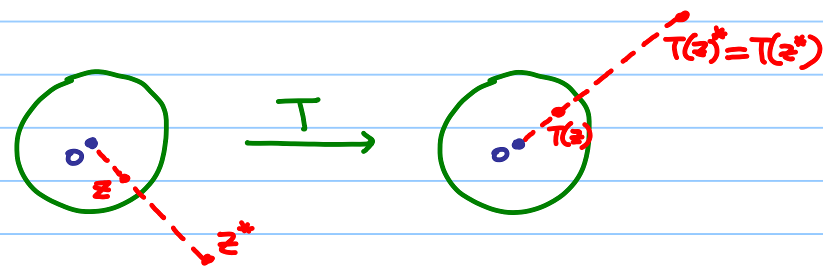
(i) all straight lines are congruent, and

(ii) two pts determine a unique straight line.

To prove this thm, we need two lemmas:

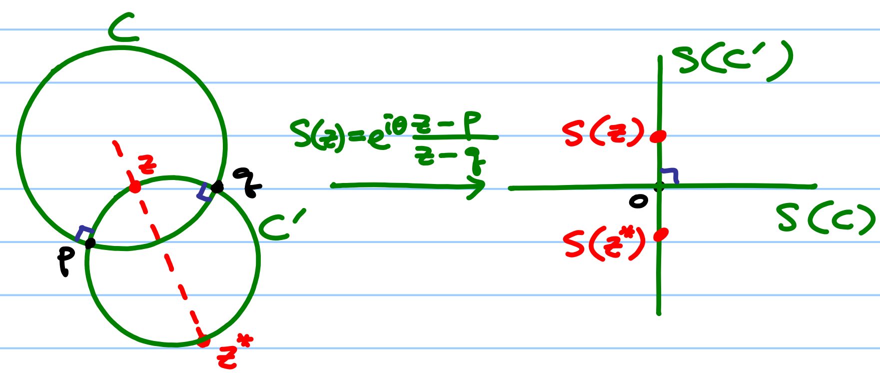
Lemma 1 Each transformation $T \in H$ maps a pair of pts symmetric w.r.t. the unit circle $\partial \mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$ to another such pair.

Pf: This is because $T(z^*) = T(z)^*$ for any $T \in H$. #



Lemma 2 Let C be a cline, and let z, z^* be symmetric w.r.t. C. Then any cline C' which is orthogonal to C and passes through z must also passes through z^*.
Conversely, any cline C' which passes through both z and z^* is orthogonal to C.

Pf: Transforms according to the figures on the right (by choosing a suitable θ). Both statements are clear on the RHS; those on the LHS follows by pulling back via S. #



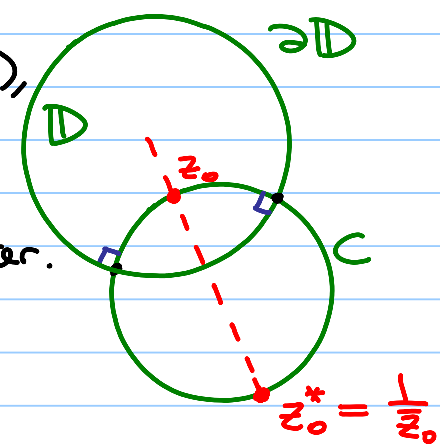
Pf of the Thm :

(i) Let C be a hyperbolic straight line, i.e. C is a cline in $\hat{\mathbb{C}}$ orthogonal to the unit circle $\partial\mathbb{D}$.

We want to show that C is congruent to the x-axis.

To do so, consider a pt $z_0 \in C \cap \mathbb{D}$.

Lemma 2 tells us that $z_0^* \in C$ as well (but $z_0^* \notin \mathbb{D}$), where z_0, z_0^* are symmetric w.r.t. $\partial\mathbb{D}$.



Let $T(z) = e^{i\theta} \dfrac{z - z_0}{1 - \bar{z}_0 z}$ for some θ to be fixed later.

Then $T \in H$ and

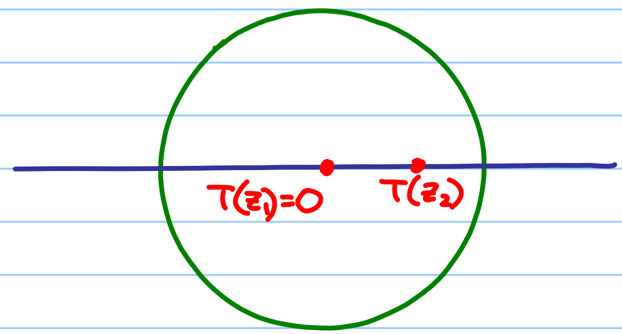
$$T(z_0) = 0 \quad \text{and} \quad T(z_0^*) = T(1/\bar{z}_0) = \infty$$

$\Rightarrow T(C)$ is a cline passing thru 0 & ∞, and orthogonal to $\partial\mathbb{D}$.
We see that $T(C) \cap \mathbb{D}$ must be a diameter of $\partial\mathbb{D}$.
By suitably choosing $\theta \in \mathbb{R}$, we have $T(C) = x$-axis.
This shows that any hyperbolic straight line is congruent to the x-axis, and hence all hyperbolic straight lines are congruent.

(ii) Let $z_1, z_2 \in \mathbb{D}$ be two distinct pts.
Then $T(z) := e^{i\theta} \dfrac{z - z_1}{1 - \bar{z}_1 z}$ takes z_1 to 0.
Choosing $\theta = -\arg\left(\dfrac{z_2 - z_1}{1 - \bar{z}_1 z_2}\right)$, we have

$$T(z_2) = e^{i\theta} \frac{z_2 - z_1}{1 - \bar{z}_1 z_2} = \left| \frac{z_2 - z_1}{1 - \bar{z}_1 z_2} \right| > 0 \quad (\text{actually } 0 < T(z_2) < 1).$$

Now the x-axis is the unique hyperbolic straight line passing thru 0 and $T(z_2)$. Hence $T^{-1}(x\text{-axis})$ is the unique hyperbolic straight line passing thru z_1 and z_2. #



Euclid's Postulates

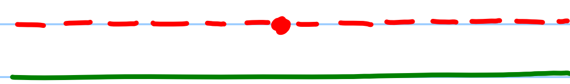
Postulate 1 Two points determine a straight line.

Postulate 2 A line can be extended indefinitely in either direction.

Postulate 3 A circle can be described with any center and radius.

Postulate 4 All right angles are congruent.

Postulate 5 Through a point not on a line, there is a unique line parallel to the given line.



Def (i) The points on the unit circle $\partial\mathbb{D}$ are called ideal points.

(ii) Two hyperbolic straight lines are called parallel if they do not intersect inside $\mathbb{D}$ but do share one ideal point.

(iii) Two hyperbolic straight lines are called hyperparallel if they do not intersect inside $\mathbb{D}$ and do not share any ideal points.