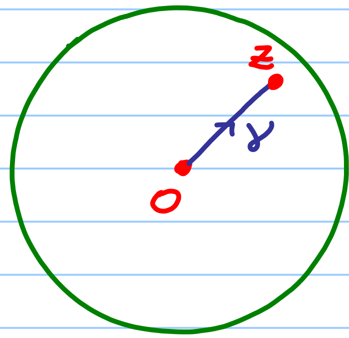
MMAT 5120 Topics in Geometry

Lecture 10

Distance formulas

We first compute the distance between 0 and $z \in \mathbb{D}$.

Claim: $d(0,z) = \ln \frac{1+|z|}{1-|z|}$



The hyperbolic straight line between 0 and z is the Euclidean line segment given by

$$\gamma : z(t) = tz \,, \quad t \in [0,1].$$

$$\Rightarrow d(0,z) = \ell(\gamma) = 2\int_0^1 \frac{|z'(t)|}{1-|z(t)|^2}\,dt = 2\int_0^1 \frac{|z|\,dt}{1-|z|^2t^2} = 2\int_0^{|z|} \frac{ds}{1-s^2}$$

$$= \int_0^{|z|} \left(\frac{1}{1-s} + \frac{1}{1+s}\right) ds = \ln \frac{1+|z|}{1-|z|}.$$

For two general points $z_1, z_2 \in \mathbb{D}$, we consider

$$T(z) = e^{i\theta} \frac{z - z_1}{1 - \bar{z}_1 z} \quad (\text{which } \theta \text{ to choose doesn't matter})$$

Then $T(z_1) = 0$, and by invariance of the hyperbolic length, we have

$$d(z_1, z_2) = d(\cancel{T(z_1)}^{\,0}, T(z_2))$$

$$= \ln \frac{1 + |T(z_2)|}{1 - |T(z_2)|}$$

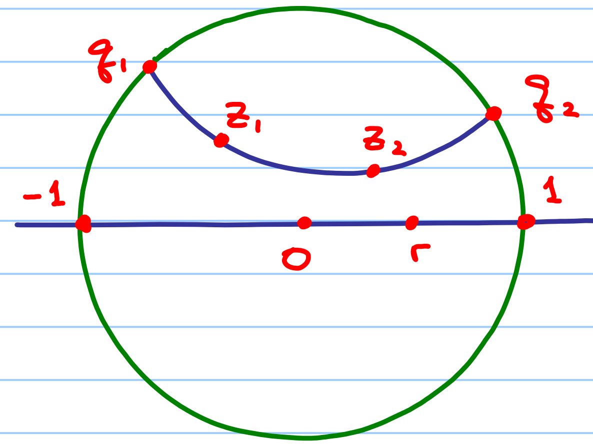
So we arrive at the formula

$$\boxed{d(z_1, z_2) = \ln \frac{1 + \dfrac{|z_2 - z_1|}{|1 - \bar{z}_1 z_2|}}{1 - \dfrac{|z_2 - z_1|}{|1 - \bar{z}_1 z_2|}}}$$

By choosing an appropriate θ, we can arrange that

$$T(z_1) = 0, \quad T(z_2) = r \in \mathbb{R}, \quad T(q_1) = -1 \text{ and } T(q_2) = 1$$

where q_1, q_2 are ideal points on ∂D, as shown in the figure on the right.



Then we have

$$\begin{aligned}(z_1, z_2, q_2, q_1) &= (T(z_1), T(z_2), T(q_2), T(q_1)) \\ &= (0, r, 1, -1) \\ &= \frac{1+r}{1-r}\end{aligned}$$

But $r = T(z_2)$, so we get another distance formula

$$\boxed{d(z_1, z_2) = \ln\,(z_1, z_2, q_2, q_1)}$$

Fundamental properties of the distance

Thm Let $z_1, z_2, z_3 \in \mathbb{D}$. Then

(1) $d(z_1, z_2) \geq 0$ and "$=$" holds iff $z_1 = z_2$.

(2) $d(z_1, z_2) = d(z_2, z_1)$.

(3) If z_1, z_2 and z_3 are collinear (in that order), then

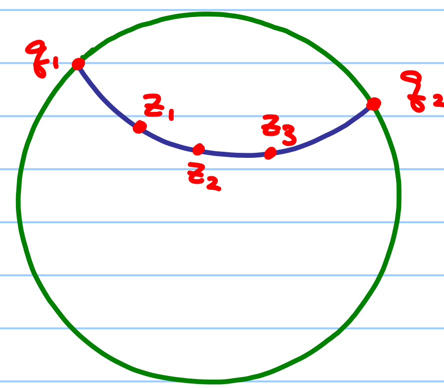
$$d(z_1, z_3) = d(z_1, z_2) + d(z_2, z_3).$$

Pf: (1) We have $\ell(\gamma) \geq 0$ since the integrand defining it is nonnegative. Since $d(z_1, z_2) = d(0, T(z_2)) = \ln \frac{1+|T(z_2)|}{1-|T(z_2)|}$, we have $d(z_1, z_2) = 0 \Leftrightarrow T(z_2) = 0 \Leftrightarrow z_1 = z_2$.

(2) This is because the length is independent of orientation. Or, we may use the formula $d(z_1, z_2) = \ln(z_1, z_2, q_2, q_1)$ and the fact that $(z_2, z_1, q_1, q_2) = (z_1, z_2, q_2, q_1)$.

(3) Applying the formula $d(z_1, z_2) = \ln(z_1, z_2, q_2, q_1)$ again, we have

$$
\begin{aligned}
& d(z_1, z_2) + d(z_2, z_3) \\
&= \ln(z_1, z_2, q_2, q_1) + \ln(z_2, z_3, q_2, q_1) \\
&= \ln\big((z_1, z_2, q_2, q_1) \cdot (z_2, z_3, q_2, q_1)\big) \\
&= \ln(z_1, z_3, q_2, q_1) \\
&= d(z_1, z_3). \quad \#
\end{aligned}
$$



Thm The shortest curve connecting two points $z_1, z_2 \in \mathbb{D}$ is given by the hyperbolic straight line segment joining z_1, z_2.

Pf: Up to a suitable transformation (as above), we may assume that $z_1 = 0$ and $z_2 = r \in (0,1)$.

Let $\gamma : z(t) = x(t) + iy(t)$, $t \in [a,b]$ be a curve joining 0 and r so that
$$\begin{cases} 0 = z_1 = z(a) = x(a) + iy(a) \\ r = z_2 = z(b) = x(b) + iy(b). \end{cases}$$

$$\Rightarrow \quad x(a) = y(a) = y(b) = 0 \quad \text{and} \quad x(b) = r.$$

Now
$$\ell(\gamma) = 2\int_a^b \frac{|z'(t)|}{1-|z(t)|^2}\,dt = 2\int_a^b \frac{\sqrt{x'(t)^2 + y'(t)^2}}{1 - x(t)^2 - y(t)^2}\,dt$$

$$\geq 2\int_a^b \frac{|x'(t)|}{1-x(t)^2}\,dt \geq 2\int_a^b \frac{x'(t)}{1-x(t)^2}\,dt$$

$$= 2\int_{x(a)}^{x(b)} \frac{ds}{1-s^2} = 2\int_0^r \frac{ds}{1-s^2} = d(0,r) = d(z_1,z_2).$$

So $\ell(\gamma) \geq \ell$(the hyperbolic straight line segment joining z_1, z_2). #

<u>Rmk</u> In fact, if $\ell(\gamma) = d(0,r)$, then we must have

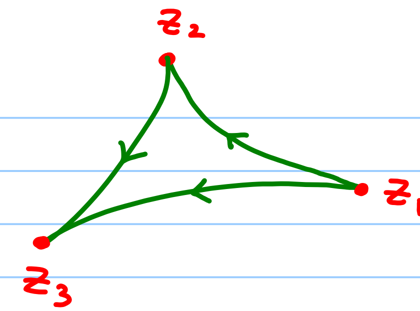
$y'(t)=0$, $y(t)=0$ and $x'(t) \geq 0$ $\forall t \in (a,b)$.

$\Longrightarrow \gamma =$ segment of x-axis going from 0 to r (up to coord. change)

$=$ hyperbolic straight line segment joining z_1, z_2.

<u>Rmk</u> The Thm (and above Rmk) are also true for piecewise smooth curves.

Cor (Triangle Inequality)
For any 3 pts $z_1, z_2, z_3 \in \mathbb{D}$, we have
$$d(z_1, z_3) \leq d(z_1, z_2) + d(z_2, z_3)$$



Pf: $d(z_1,z_2) + d(z_2,z_3) = \ell($hyperbolic line from z_1 to z_2 + hyperbolic line from z_2 to $z_3)$
$\leq \ell($hyperbolic line from z_1 to $z_3)$ by above Thm
$= d(z_1, z_3)$. #

Euclid's Postulates (revisited)

We've seen that both "Postulate 1: Two points determine a straight line" and "Postulate 4: All right angles are congruent" hold in hyperbolic geometry.

Claim: In hyperbolic geometry, Euclid's
"Postulate 2: A line can be extended indefinitely in either direction" and
"Postulate 3: A circle can be described with any center and radius"
also hold.

Pf: For P2, note that

$$\lim_{r \to 1} d(0,r) = \lim_{r \to 1} \ln \frac{1+r}{1-r} = +\infty$$

So $\forall N > 0$, $\exists\, 1 > r_1 > r$

s.t. $\ln \frac{1+r_1}{1-r_1} > \ln \frac{1+r}{1-r} + N$ or $d(0,r_1) > d(0,r) + N$

This means that hyperbolic line segments can be extended indefinitely.

For P3, using a transformation, we can always assume that the center is the origin $0 \in \mathbb{D}$.

Given any $R > 0$, we can take $r = \frac{e^R - 1}{e^R + 1}$.

Then $d(0, re^{i\theta}) = d(0, r) = \ln \frac{1+r}{1-r} = R$ for any $\theta \in \mathbb{R}$.

So the Euclidean circle centered at $0 \in \mathbb{D}$ with radius $r = \frac{e^R - 1}{e^R + 1}$ is a hyperbolic circle centered at $0 \in \mathbb{D}$ with hyperbolic radius R. #

We conclude that hyperbolic geometry is a non-Euclidean geometry in the strict sense.

Rmk We can see that a hyperbolic circle is described as the locus

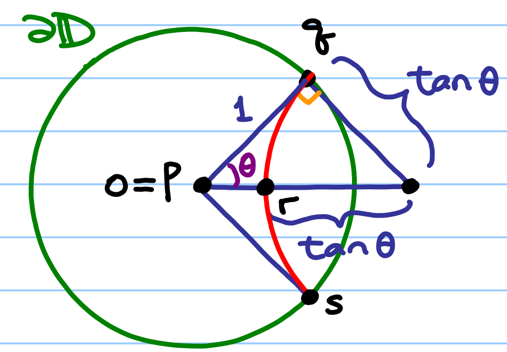
$$\{z \in \mathbb{D} : d(z, z_0) = R\}$$ for some $z_0 \in \mathbb{D}$ and $R > 0$.

Formula of Lobatchevsky

Thm Let the point p be given at the hyperbolic distance d from a hyperbolic straight line. Let θ be the angle of parallelism of p with respect to this line. Then

$$e^{-d} = \tan\frac{\theta}{2}$$

Pf: Applying a transformation $T \in H$, we can assume that $p = 0 \in \mathbb{D}$ and the given hyperbolic line is symmetrical about the x-axis, so that the perpendicular from p to the hyperbolic line is the x-axis.



Then $d = d(0,r) = \ln \frac{1+r}{1-r}$, and

$$r = \sec\theta - \tan\theta = \frac{1-\sin\theta}{\cos\theta}$$

$$\Rightarrow e^{-d} = \frac{1-r}{1+r} = \frac{1 - \frac{1-\sin\theta}{\cos\theta}}{1 + \frac{1-\sin\theta}{\cos\theta}}$$

$$= \frac{\cos\theta + \sin\theta - 1}{\cos\theta - \sin\theta + 1}$$

$$= \frac{\frac{1-t^2}{1+t^2} + \frac{2t}{1+t^2} - 1}{\frac{1-t^2}{1+t^2} - \frac{2t}{1+t^2} + 1}$$

$$= \frac{1-t^2+2t-1-t^2}{1-t^2-2t+1+t^2} = t. \ \#$$

[$\cos\theta = \frac{1-t^2}{1+t^2}$, $\sin\theta = \frac{2t}{1+t^2}$,

where $t = \tan\frac{\theta}{2}$]