MMAT 5120 Topics in Geometry

Lecture 1

§ Complex numbers

A complex number is an expression of the form

$$z = x + iy$$

where $\operatorname{Re} z := x \in \mathbb{R}$ is called the real part of z,

$\operatorname{Im} z := y \in \mathbb{R}$ is called the imaginary part of z,

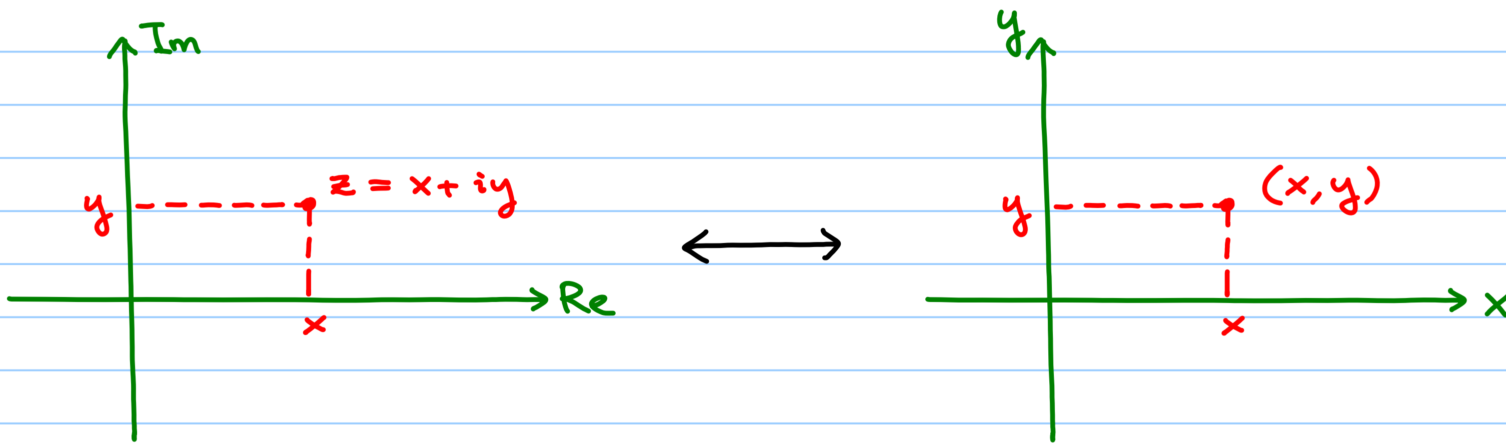
and $i = \sqrt{-1}$ is called the imaginary unit.

The set of complex numbers is denoted as

$$\mathbb{C} = \{ x + iy : x, y \in \mathbb{R} \} \quad \text{where } i = \sqrt{-1}.$$

$\rightsquigarrow$ one-to-one correspondence

$$\mathbb{C} \cong \mathbb{R}^2$$



This is why we call $\mathbb{C}$ the complex plane.

Operations on complex numbers

We have the following operations :

- For $z = x + iy$, $w = s + it \in \mathbb{C}$, we define

(Addition) $z + w := (x+s) + i(y+t)$

(Multiplication) $z \cdot w := (xs - yt) + i(ys + xt)$

(Since $i^2 = -1$,
$(x+iy)(s+it)$
$= xs + iys + ixt + \cancel{i^2}^{-1} yt$)

Not hard to check

commutativity	$z + w = w + z$	$z \cdot w = w \cdot z$
associativity	$(z+w)+u = z + (w+u)$	$(z \cdot w) \cdot u = z \cdot (w \cdot u)$
exist. of id.	$z + 0 = z \quad \forall z \in \mathbb{C}$	$z \cdot 1 = z \quad \forall z \in \mathbb{C}$
exist. of inv.	$\forall z \in \mathbb{C}, \exists -z \in \mathbb{C}$ s.t. $z + (-z) = 0$	$\forall z \in \mathbb{C} \setminus \{0\}, \exists z^{-1} \in \mathbb{C}$ s.t. $z \cdot z^{-1} = 1$

If $z = x + iy$
then
$$z^{-1} = \frac{x - iy}{x^2 + y^2}$$

distributive law & $z \cdot (w + u) = z \cdot w + z \cdot u$ $\rightsquigarrow$ $(\mathbb{C}, +, \cdot)$ is a field (like $\mathbb{R}$)

- For $z = x + iy \in \mathbb{C}$, we define its modulus as

$$|z| := \sqrt{x^2+y^2} \in \mathbb{R}_{\geq 0}$$

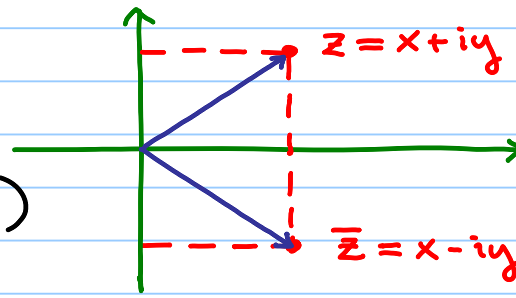
= length of the vector $(x,y) \in \mathbb{R}^2$

= distance between (x,y) and $(0,0)$

and its conjugate as

$$\bar{z} := x - iy \in \mathbb{C}$$

(= reflection of z along the real axis)



Basic properties :

1) $\bar{\bar{z}} = z$

2) $\overline{z+w} = \bar{z} + \bar{w}$,
$\overline{z \cdot w} = \bar{z} \cdot \bar{w}$,
$\overline{z^{-1}} = \bar{z}^{-1}$

3) $\mathrm{Re}\, z = \dfrac{z+\bar{z}}{2}$, $\mathrm{Im}\, z = \dfrac{z-\bar{z}}{2i}$

4) $|z| = |\bar{z}|$

5) $|z|^2 = z\bar{z}$

6) $|zw| = |z||w|$

7) $\begin{cases} \mathrm{Re}\, z \leq |\mathrm{Re}\, z| \leq |z| \\ \mathrm{Im}\, z \leq |\mathrm{Im}\, z| \leq |z| \end{cases}$

8) (Triangle Inequality)

$$|z+w| \leqslant |z| + |w|$$

and "=" iff $z \parallel w$ i.e. $\exists\, a, b \in \mathbb{R}$ not both zero s.t. $az = bw$.

Pf:

$$\begin{aligned} |z+w|^2 &= (z+w)(\bar{z}+\bar{w}) \\ &= z\bar{z} + z\bar{w} + \bar{z}w + w\bar{w} \\ &= |z|^2 + 2\,\mathrm{Re}(z\bar{w}) + |w|^2 \\ &\leqslant |z|^2 + 2|z||w| + |w|^2 = (|z|+|w|)^2. \end{aligned}$$

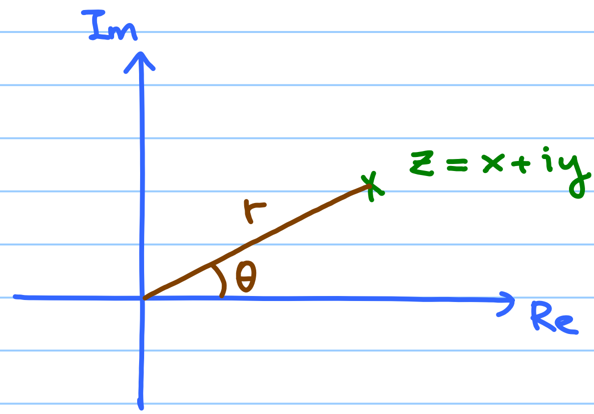
$$\begin{aligned} \text{Equality holds} &\iff \mathrm{Re}(z\bar{w}) = |z\bar{w}| \\ &\iff z\bar{w} \in \mathbb{R} \\ &\iff z \parallel w. \qquad \# \end{aligned}$$

Polar coordinates

Cartesian coordinates $\longleftrightarrow$ Polar coordinates

(x, y) $\quad\quad$ (r, θ)

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \begin{cases} r = \sqrt{x^2+y^2} \\ \tan\theta = \dfrac{y}{x} \end{cases}$$



$\rightsquigarrow \quad z = x + iy = r(\cos\theta + i\sin\theta)$

- $r = |z|$
- θ is undefined for $z=0$
- For $z \neq 0$, θ is defined only up to $2k\pi$ for $k \in \mathbb{Z}$; each value of θ s.t. $z = |z|(\cos\theta + i\sin\theta)$ is called an argument of z.

- We set $\arg z :=$ set of all arguments of $z \in \mathbb{C} \setminus \{0\}$.
- The principal argument of z, denoted as $\operatorname{Arg} z$, is the argument of z lying in $(-\pi, \pi]$, i.e., $-\pi < \operatorname{Arg} z \leq \pi$.

So we have $\arg z = \{ \operatorname{Arg} z + 2k\pi : k \in \mathbb{Z} \}$

Euler's formula

$$e^{i\theta} := \cos\theta + i \sin\theta$$

(justification: by Taylor series

$$e^{i\theta} = 1 + i\theta + \frac{1}{2!}(i\theta)^2 + \frac{1}{3!}(i\theta)^3 + \frac{1}{4!}(i\theta)^4 + \frac{1}{5!}(i\theta)^5 + \cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right)$$

$$= \cos\theta + i \sin\theta$$)

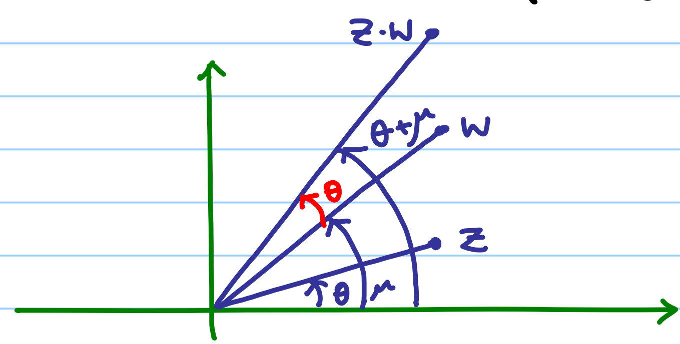
So now we have $z = re^{i\theta}$, and

$$e^{i\theta} \cdot e^{i\mu} = e^{i(\theta + \mu)}$$ by compound angle formula

$\Rightarrow$ (de Moivre's Thm) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i \sin n\theta$ as $(e^{i\theta})^n = e^{in\theta}$

For $z = |z|e^{i\theta}$, $w = |w|e^{i\mu} \in \mathbb{C}$, we have

$$\begin{cases} z \cdot w = |z||w| e^{i(\theta+\mu)} \\ \dfrac{z}{w} = \dfrac{|z|}{|w|} e^{i(\theta-\mu)} \end{cases} \quad \text{or} \quad \begin{cases} |z \cdot w^{\pm 1}| = |z||w|^{\pm 1} \\ \arg(z \cdot w^{\pm 1}) = \arg z \pm \arg w \end{cases}$$



§ Geometric transformations

A transformation is a one-to-one, onto (i.e. bijective) function whose image and domain are the same set.

Here are some examples:

- Translations For a fixed $b \in \mathbb{C}$, the translation

$$f_b : \mathbb{C} \longrightarrow \mathbb{C}, \quad z \longmapsto z + b$$

is a transformation.

Pf: $f_b(z) = f_b(w) \Rightarrow z + b = w + b \Rightarrow z = w$, so f_b is 1-1.

$\forall w \in \mathbb{C}$, we have $f_b(w-b) = (w-b) + b = w$, so f_b is onto. #

Actually, the inverse of f_b is $(f_b)^{-1} = f_{-b}$, also a translation.

- Rotations For a fixed $\theta \in \mathbb{R}$, the rotation
$$g_\theta : \mathbb{C} \longrightarrow \mathbb{C}, \quad z \longmapsto e^{i\theta} z$$
is a transformation, with inverse $(g_\theta)^{-1} = g_{-\theta}$. (Exercise)

- Homothetic transformations (stretching or shrinking)
For $k \in \mathbb{R}_{>0}$, the scaling
$$s_k : \mathbb{C} \longrightarrow \mathbb{C}, \quad z \longmapsto kz$$
is a transformation, with inverse $(s_k)^{-1} = s_{1/k}$. (Exercise)

- Inversion $T : \mathbb{C}^* := \mathbb{C} \setminus \{0\} \longrightarrow \mathbb{C}^*, \quad z \longmapsto \frac{1}{z}$

is a transformation, called the inversion, whose inverse is itself.