THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MMAT 5120 Topics in Geometry 2021-22
Lecture 11 practice problems solution
6th April 2022

- The practice problems are meant as exercise to the students. You are **NOT** required to submit your solutions, but you are encouraged to work through all of them in order to understand the course materials. The problems will be uploaded on Fridays and solutions will be uploaded on Wednesdays before the next lecture.

- Please send an email to **echlam@math.cuhk.edu.hk** if you have any questions.

1. We have to show that $\bar{H}$ preserves $\mathbb{U}$ and also that it is a transformation group. For the former, note that $\frac{az+b}{cz+d} \in \bar{H}$ clearly sends $\mathbb{R}$ to $\mathbb{R}$, as all the entries are real. We have to show that if $\mathrm{Im}(z) > 0$ then so is $\mathrm{Im}(\frac{az+b}{cz+d}) > 0$. Now consider

$$\begin{aligned}\mathrm{Im}\left(\frac{az+b}{cz+d}\right) &= \mathrm{Im}\left(\frac{(az+b)(c\bar{z}+d)}{(cz+d)(c\bar{z}+d)}\right)\\ &= \mathrm{Im}\left(\frac{ac|z|^2+bd+adz+bc\bar{z}}{|cz+d|^2}\right)\\ &= \frac{\mathrm{Im}(adz-bcz+bcz+bc\bar{z})}{|cz+d|^2}\\ &= \frac{(ad-bc)\mathrm{Im}(z)}{|cz+d|^2} > 0\end{aligned}$$

Where in the above, the red colored parts are all real and hence their real parts are 0. And $ad-bc > 0$ by definition.

Now to show that $\bar{H}$ is a transformation group, we need to check that the identity element is in $\bar{H}$ (obvious), the inverse of a transformation in $\bar{H}$ is also in $\bar{H}$, as well as composition of two transformations in $\bar{H}$ is again in $\bar{H}$. One can see that both are true if one is comfortable with working with matrices. These just follow from the fact that inverse and composition of real matrices are again real.

2. (a) $(\mathbb{D}, H)$ and $(\mathbb{U}, \bar{H})$ are isomorphic geometries. As we have seen from the lecture, there is an explicit transformation $S(z) = i\frac{1+z}{1-z}$ that takes $\mathbb{D}$ to $\mathbb{U}$. This means that we have a bijection (isomorphism) between H and $\bar{H}$ as well. If $T \in H$, then $STS^{-1} \in \bar{H}$. Notice that conjugating an element S does not change the type of transformation T is. If T is elliptic, hyperbolic or parabolic then STS^{-1} is again elliptic, hyperbolic or parabolic respectively, and vice versa. So the classification for H and $\bar{H}$ are exactly the same.

   If you want more details, the reason is that if say T has two fixed points and $RTR^{-1}(z) = \lambda z$ is the normal form for T. One can simply take $RTR^{-1} = (RS^{-1})STS^{-1}(RS^{-1})^{-1} = \lambda z$ and it would be the normal form for STS^{-1}. Clearly the λ is the same. The argument is similar for T with one fixed point.

(b) Following directly from lecture 7 practice problem Q2, we have T is elliptic if it has one fixed point in $\mathbb{U}$, T is hyperbolic if it has two fixed points on $\mathbb{R} \cup \{\infty\} = \partial\mathbb{U}$, and T is parabolic if it has a unique fixed point on $\mathbb{R} \cup \{\infty\}$.

(c) We can take the conditions in part (b) and study what it means in terms of fixed point formula $f(z) = z$. Let's simplify a bit first, $\frac{az+b}{cz+d} = z$ so $cz^2 + (d-a)z - b = 0$. The solutions are given by $\frac{a-d\pm\sqrt{(d-a)^2-4bc}}{2c}$. Now the good news is all entries a, b, c, d are real. The first situation is if $c = 0$, then the unique solution is ∞, this falls under the parabolic case. Now assume $c \neq 0$, and $(d-a)^2 - 4bc = 0$, then there is a unique real solution, so it is again the parabolic case.

Now if $(d-a)^2 - 4bc > 0$, there will be two real solutions, so there are two fixed points on the real line, this is the hyperbolic case.

Finally if $(d-a)^2 - 4bc < 0$, there are two non-real solutions that are complex conjugate to each other, so there is a fixed point in $\mathbb{U}$. This is the elliptic case.

3. Like we said above, the geometric statements that are true in $\mathbb{D}$ are also true in $\mathbb{U}$. Since horocycles remain to be horocycle under the transformation $S : \mathbb{D} \to \mathbb{U}$, and by homework 2 there are two horocycles passing through every pair of points, it must be the case for $\mathbb{U}$ as well. The first horocycle is not hard to find, it is just the cycle centered at i with radius 1. The second horocycle requires a bit of creative thinking. It is actually just the horizontal line $\text{Im}(z) = 1$ (union with ∞). This line is tangent to $\mathbb{R} \cup \{\infty\} = \partial\mathbb{U}$ at ∞. This is perhaps not obvious from the complex plane, but you will see it immediately if you draw the cline $\text{Im}(z) = 1$ in the extended complex plane $\hat{\mathbb{C}}$.

   Of course, it is not very satisfying to resort to a "just look at the picture yourself" argument. Here is a more rigorous proof. Consider the image of the first horocycle, i.e. the circle centered at i with radius 1, under the transformation $-\frac{2}{z} \in \bar{H}$. Clearly the image of horocycle under a transformation in $\bar{H}$ is again a horocycle. And notice that $\frac{-2}{1+i} = i - 1$ and $\frac{-2}{i-1} = 1 + i$. So the resulting horocycle is actually one that passes through the two desired points. Now $\frac{-2}{0} = \infty$. So the transformation takes the horocycle to the straight line passing through $1 + i, i - 1$. This concludes the proof.

4. Since we define distance in $\mathbb{U}$ by taking the distance of corresponding map under $S^{-1} : \mathbb{U} \to \mathbb{D}$. In particular S^{-1} is a Mobius transform and would preserve cross ratio. And since S^{-1}, S also take hyperbolic straight lines to hyperbolic straight lines. We just have to find the hyperbolic straight line connected ri, si, which is clearly the positive imaginary axis. The two ideal points are $0, \infty$. So $d(ri, si) = \ln(ri, si, \infty, 0)$ for $r < s$. This computes to be $\ln \frac{(ri-\infty)(si-0)}{(ri-0)(si-\infty)} = \ln \frac{s}{r}$.

5. We can start from a random vertex, and pick another vertex that is one vertex apart, connect the two points using a hyperbolic straight line. Now we have divide a hyperbolic n-gon into a hyperbolic triangle and a hyperbolic $n-1$ gon. This process can be repeated until all the polygons are hyperbolic triangles. This process is known as a triangulation. In general, an n-gon can be triangulated into $n-2$ hyperbolic triangles. Clearly the sum of their areas is just the total area of the n-gon. So the area $A = (n-2)\pi - \sum_i \theta_i$ where $\sum \theta_i$ denotes the sum of internal angles.