THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MMAT 5120 (2021-22, Term 2)
Topics in Geometry
Homework 2
Due Date: 24th March 2022

We denote by $\mathbf{i}$ the imaginary unit $\sqrt{-1}$, by $\mathbf{M}$ the group of Möbius transformations, and by $\mathbf{H}$ the group of hyperbolic transformations.

1. Let z_1, z_2, z_3 be distinct points on $\hat{\mathbb{C}}$, and w be any point on $\hat{\mathbb{C}}$. Show that there exists $z \in \hat{\mathbb{C}}$ such that $(z, z_1, z_2, z_3) = w$.

2. Let
$$\frac{1}{T(z) - p} = \frac{1}{z - p} + \beta$$
be the normal form of a parabolic transformation whose fixed point p is not ∞. Show that
$$\beta = -\frac{1}{z_0 - p} = \frac{1}{T(\infty) - p},$$
where z_0 is the point such that $T(z_0) = \infty$.

3. Consider the Möbius transformation $T \in \mathbf{M}$ defined by
$$T(z) = \frac{z}{z - \mathbf{i}}.$$
   (a) Find the fixed point(s) of T.
   (b) Find the normal form of T, hence deciding what type of transformation it is.
   (c) Sketch the appropriate coordinate system of Steiner circles, and use arrows to indicate the motion of T.

4. Show directly that a hyperbolic transformation $T \in \mathbf{H}$ given by
$$T(z) = e^{\mathbf{i}\theta} \frac{z - z_0}{1 - \overline{z}_0 z},$$
where $|z_0| < 1$ and $\theta \in \mathbb{R}$, indeed maps the open unit disk $\mathbb{D}$ into itself, i.e. $|T(z)| < 1$ when $|z| < 1$.

5. Show that a Möbius transformation of the form
$$T(z) = \frac{az + b}{\overline{b}z + \overline{a}},$$
where $|a|^2 - |b|^2 = 1$, is a hyperbolic transformation, and conversely, any hyperbolic transformation $T \in \mathbf{H}$ is of this form.