Lecture 1: Vector spaces

Field

Definition: A field is a set F along with two binary operations: $+$ (addition) and $\cdot$ (multiplication) such that:

- For $\forall\, x, y \in F$, $x + y = y + x$ and $x \cdot y = y \cdot x$
- For $\forall\, x, y, z \in F$, $(x+y)+z = x+(y+z)$ and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$
- For $\forall x, y, z \in F$, $x \cdot (y+z) = x \cdot y + x \cdot z$
- $\exists!$ element $0 \in F \ni \forall\, x \in F$, $x + 0 = x$
- $\exists!$ element $1 \in F \ni \forall\, x \in F$, $x \cdot 1 = x$
- For $\forall\, x \in F$, $\exists$ an element $(-x) \in F \ni x + (-x) = 0$
- For $\forall\, x \in F$ (excluding $x = 0$), $\exists$ an element $x^{-1} \in F \ni x \cdot x^{-1} = 1$

Remark:
- We often write xy for $x \cdot y$
- If F is finite, we say it is a finite field

Examples of field

$$\left.\begin{array}{ll} 1. & F = \mathbb{R} \\ 2. & F = \mathbb{C} \end{array}\right\} \text{Most often considered in Math 2048.}$$

3. $F = \{\text{Rational numbers}\} = \{ p/q : p, q \in \mathbb{Z} \}$

4. Finite field of order p (where p is a prime number)

Define $F_p = \{0, 1, 2, \ldots, p-1\}$ and $+ / \cdot$ are defined as:

$+$: for $\forall x, y \in F_p$, $x + y$ are performed modulo p.
That is, $x + y$ is the remainder of $(x+y)/p$

$\cdot$: for $\forall x, y, \in F_p$, $x \cdot y$ is the remainder of $x \cdot y / p$.

$F_2 = \{0, 1\}$ is the binary field (important for information theories)

Vector Space

Goal: Build an abstract space (space of objects) simulating $\mathbb{R}^n$ or $\mathbb{C}^n$ (with addition and multiplication/scaled)

Definition: A vector space over F is a set V equipped w/ two operations:

(addition) $+ : V \times V \to V$, $\left(\overset{\overset{V}{\downarrow}}{\vec{x}}, \overset{\overset{V}{\downarrow}}{\vec{y}}\right) \mapsto \vec{x}+\vec{y} \in V$

(Scalar multiplication) $\cdot : F \times V \to V$, $\left(\underset{\underset{F}{\uparrow}}{a}, \underset{\underset{V}{\uparrow}}{\vec{x}}\right) \mapsto a\vec{x} \in V$

satisfying 8 properties:

$$+\left\{\begin{array}{lll}
(VS1): & \vec{x}+\vec{y}=\vec{y}+\vec{x} & \forall \vec{x},\vec{y}\in V \\
(VS2): & (\vec{x}+\vec{y})+\vec{z}=\vec{x}+(\vec{y}+\vec{z}) & \forall \vec{x},\vec{y},\vec{z}\in V \\
(VS3): & \exists\ \vec{0}\in V \text{ s.t. } \vec{x}+\vec{0}=\vec{x} & \forall \vec{x}\in V \\
(VS4): & \forall \vec{x}\in V,\ \exists\ \vec{y}\in V \text{ s.t. } \vec{x}+\vec{y}=\vec{0} & (\text{inverse})
\end{array}\right.$$

$$\cdot\left\{\begin{array}{lll}
(VS5): & \underset{\substack{\uparrow \\ F}}{1}\,\vec{x}=\vec{x} & \forall \vec{x}\in V \\
(VS6): & (\underset{\substack{\uparrow \\ F}}{a}\,\underset{\substack{\uparrow \\ F}}{b})\vec{x}=a(b\vec{x}) & \forall a,b\in F,\ \forall \vec{x}\in V
\end{array}\right.$$

$$+/\cdot\left\{\begin{array}{lll}
(VS7): & \underset{\substack{\uparrow \\ F}}{a}\,(\underset{\substack{\uparrow \\ V}}{\vec{x}}+\underset{\substack{\uparrow \\ V}}{\vec{y}})=a\vec{x}+a\vec{y} & \forall a\in F,\ \forall \vec{x},\vec{y}\in V \\
(VS8): & (a+b)\vec{x}=a\vec{x}+b\vec{x} & \forall a,b\in F,\ \forall \vec{x}\in V
\end{array}\right.$$

Remark: an element in F is called scalar.
an element in V is called vector.

Examples of vector spaces

- $F^n = \{(x_1, x_2, \dots, x_n) : x_j \in F$ for $j = 1, 2, \dots, n\}$ w/

  $(x_1, x_2, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n)$

  $a(x_1, \dots, x_n) = (ax_1, ax_2, \dots, ax_n)$

- $M_{m\times n}(F) = \{ m \times n$ matrices w/ entries in $F\}$
  w/ matrix addition and scalar multiplication

- $P(F) = \{$ polynomials w/ coefficients in $F\}$
  w/ polynomial addition and scalar multiplication.

- $F^\infty = \{(x_1, x_2, \dots) : x_j \in F, j = 1, 2, \dots\}$
  w/ component-wise addition and scalar multiplication.

- $\mathrm{Sym}_{n\times n}(F) = \{\, n \times n$ $\overset{\text{symmetric}}{\text{matrices}}$ A w/ entries in F : $A^T = A \}$

- Let S be any non-empty set.

Then: $\mathcal{F}(S, F) = \{$ functions $f : S \to F \}$

is a vector space over F under:

$$(\underset{\mathcal{F}(S,F)}{\underset{\wedge}{f}} + \underset{\mathcal{F}(S,F)}{\underset{\wedge}{g}})(\overset{\in S}{s}) \overset{\text{def}}{=} f(s) + g(s) \; ; \qquad (\underset{F}{\underset{\wedge}{a}} f)(s) \overset{\text{def}}{=} a f(s).$$

- $\mathbb{C}$ is a vector space over $F = \mathbb{C}$

<u>Question</u>: Is $V = \mathbb{R}$ a vector space over $F = \mathbb{C}$??

- Consider the differential equation:

$$(*) \quad \frac{d^2y}{dX^2} + a\frac{dy}{dx} + by = 0 \qquad (a, b \in \mathbb{R})$$

Let S be the set of twice differentiable functions on $\mathbb{R}$ satisfying $(*)$.

Then S is a vector space under usual addition and scalar multiplication is a vector space.

Proposition: Let V be a vector space over F. Then:

(a) The element $\vec{0}$ in (VS3) is unique, called zero vector

(b) $\forall \vec{x} \in V$, the element $\vec{y}$ in (VS4) is unique, called the additive inverse (Denoted as $-\vec{x}$)

(c) $\vec{x} + \vec{z} = \vec{y} + \vec{z} \Rightarrow \vec{x} = \vec{y}$ (Cancellation law)

(d) $\underset{\underset{F}{\uparrow}}{0}\,\vec{x} = \vec{0} \quad \forall \vec{x} \in V$

(e) $\underset{\underset{F}{\uparrow}}{(-a)}\,\vec{x} = -(a\vec{x}) = a(-\vec{x}), \quad \forall a \in F, \quad \forall \vec{x} \in V$

(f) $\underset{\underset{F}{\uparrow}}{a}\,\vec{0} = \vec{0} \quad \forall a \in F$

Subspace

Definition: A subset W of a vector space V over a field F is called a subspace of V if W is a vector space over F under the same addition and scalar multiplication inherited from V.

Proposition: Let V be a vector space V over F. A subset $W \subset V$ is a subspace iff the following 3 conditions hold:

(a) $\vec{0}_v \in W$

(b) $\vec{x} + \vec{y} \in W$, $\forall \vec{x}, \vec{y} \in W$ (closed under $+$)

(c) $a\vec{x} \in W$, $\forall a \in F$, $\forall \vec{x} \in W$ (closed under $\cdot$)

Examples:

- For any vector space V / F,

  $\{\vec{0}\} \subset V$; $V \subset V$ (trivial subspaces)

  $\overset{''}{\text{zero}}$ subspace

- For $V = M_{n\times n}(F)$,

  $W_1 = \{\text{ diagonal matrices }\} \subset V$ subspace

  $W_2 = \{ A \in M_{n\times n}(F) : \det(A) = 0 \} \subset V$

  $W_3 = \{ A \in M_{n\times n}(F) : \operatorname{tr}(A) = 0 \} \subset V$

  subspace

- For $V = P(F)$

$$P_n(F) \overset{\text{def}}{=} \{ f \in P(F) : \deg(f) \leq n \}$$ is a subspace

$$W \overset{\text{def}}{=} \{ f \in P(F) : \deg(f) = n \}$$

- Consider $V = F^n = \{ \underset{\overset{\shortparallel}{\vec{x}^T}}{(x_1, x_2, \ldots, x_n)} : x_j \in F$ for $j = 1, 2, \ldots, n \}$

Consider linear system:

$$\left\{ \begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \cdots \quad\quad + a_{2n}x_n &= b_2 \\ &\;\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \right. \iff A\vec{x} = \vec{b}$$

gives a subset, the solution set $S \subset V$

Is S a subspace?

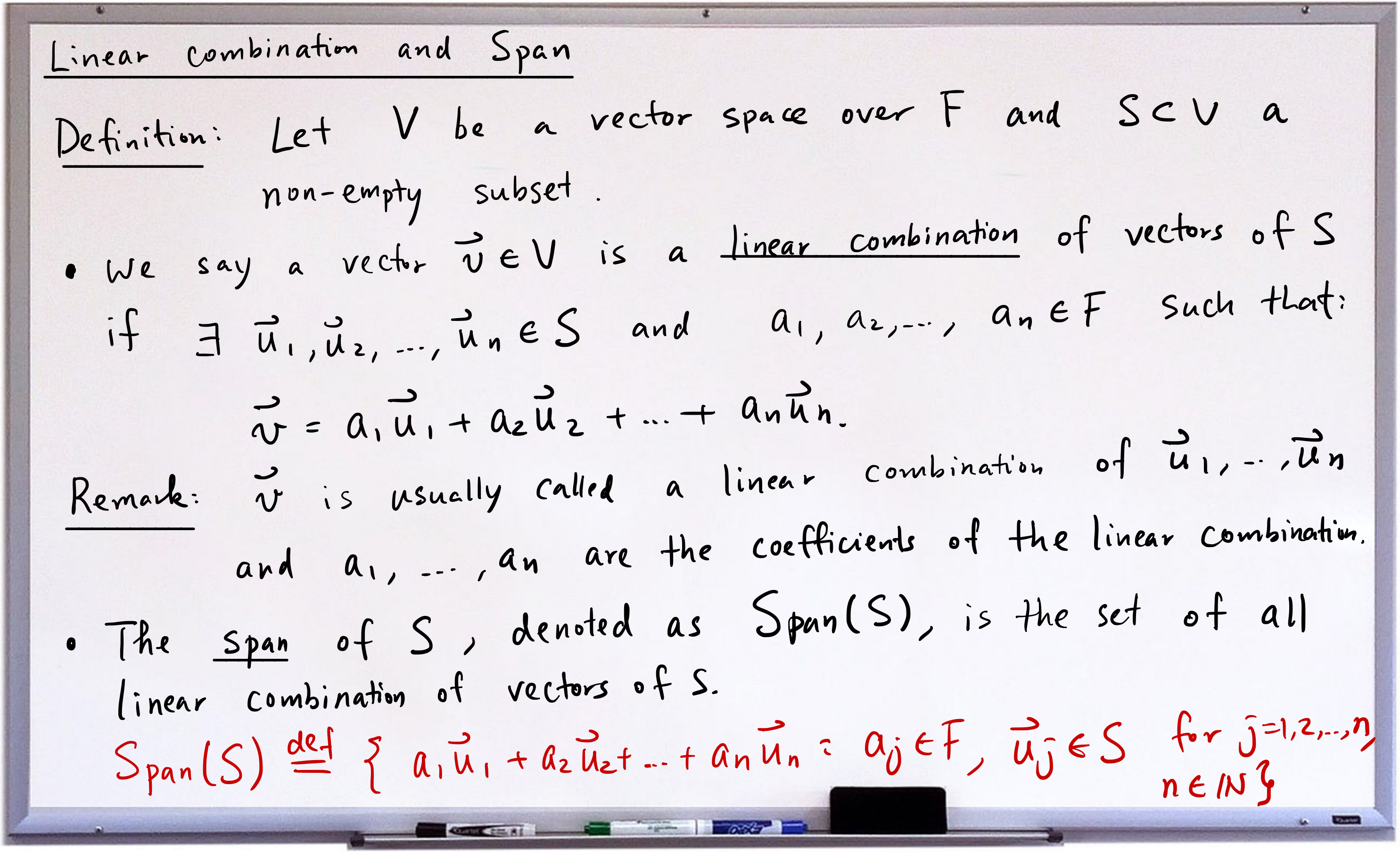
Theorem: Any intersection of subspaces of a vector space V is a subspace of V.

Question: $\underset{\underset{V}{\cap}}{W_1}$ = subspace ; $\underset{\underset{V}{\cap}}{W_2}$ = subspace

$\Downarrow$

$W_1 \cap W_2$ is subspace

Is $\underset{\underset{V}{\cap}}{W_1 \cup W_2}$ a subspace ??



Linear combination and Span

Definition: Let V be a vector space over F and $S \subset V$ a non-empty subset.

- We say a vector $\vec{v} \in V$ is a <u>linear combination</u> of vectors of S if $\exists\ \vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n \in S$ and $a_1, a_2, \ldots, a_n \in F$ such that:

$$\vec{v} = a_1 \vec{u}_1 + a_2 \vec{u}_2 + \ldots + a_n \vec{u}_n .$$

Remark: $\vec{v}$ is usually called a linear combination of $\vec{u}_1, \ldots, \vec{u}_n$ and $a_1, \ldots, a_n$ are the coefficients of the linear combination.

- The <u>span</u> of S, denoted as $\operatorname{Span}(S)$, is the set of all linear combination of vectors of S.

$$\operatorname{Span}(S) \overset{def}{=} \{ a_1 \vec{u}_1 + a_2 \vec{u}_2 + \ldots + a_n \vec{u}_n : a_j \in F,\ \vec{u}_j \in S \text{ for } j = 1, 2, \ldots, n,\ n \in \mathbb{N} \}$$

Remark: • By convention, $\operatorname{span}(\underset{\underset{\text{empty set}}{\shortparallel}}{\phi}) \overset{\text{def}}{=} \{\vec{0}\}$.

e.g. $1 \in \operatorname{Span}(\{1+x^2, 1-x^2\})$

x

Example:

- $F^n = \text{Span}(\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\})$ where $\vec{e}_j = (0, 0, \dots, \underset{\underset{j\text{th}}{\uparrow}}{1}, 0, \dots 0)$

- $P(F) = \text{Span}(\{1, x, x^2, \dots, x^n, \dots\})$

- $M_{n\times n}(F) = \text{Span}(S)$

$$S = \left\{ E_{ij} \overset{def}{=} \begin{array}{c|cccccc|} & & & & \downarrow j\text{th} & & \\ & 0 & 0 & \cdots & - & - & 0 \\ & 0 & - & - & - & & 0 \\ i\text{th} \rightarrow & 0 & \cdots & - & 1 & & 0 \\ & & & & & & \\ & 0 & - & \cdots & & & 0 \end{array} : 1 \le i, j \le n \right\}$$

- Given $\vec{u}_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix}, \vec{u}_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{pmatrix}, \dots, \vec{u}_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{pmatrix}$

Then: $\underset{= \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}}{\vec{v}} \in \text{Span}(\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n\})$ iff

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = v_1 \\ a_{21}x_1 + \quad \cdots \quad + a_{2n}x_n = v_2 \\ \quad \vdots \\ a_{n1}x_1 + \quad \cdots \quad + a_{nn}x_n = v_n \end{cases}$$

has a sol.

Theorem: Let $S \subset V$ be a subset of a vector space V over F.

Then, $\text{span}(S)$ is the smallest ① subspace ② of V consisting S.

(If W is a subspace containing S, then $\text{Span}(S) \subset W$)

Linear independence

Definition: Let V be a vector space over F. A subset $S \subset V$ is said to be linearly dependent if ∃ distinct $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n \in S$ and scalars $a_1, a_2, \ldots, a_n \in F$, not all zero, s.t.

$$a_1\vec{u}_1 + a_2\vec{u}_2 + \cdots + a_n\vec{u}_n = \vec{0}$$

Otherwise, it is said to be linearly independent.

e.g.

- The empty set $\phi \subset V$ is linearly independent.
- If $\vec{0} \in S$, the S is linearly dependent $\left(\underset{\overset{\uparrow}{S}}{5}\,\vec{0} = \vec{0} \right)$
- If $S = \{\vec{u}\}$ and $\vec{u} \neq \vec{0}$, then S is linearly independent.

$$\left(\begin{array}{c} \lambda \overset{\neq \vec{0}}{\vec{u}} = \vec{0} \\ \Rightarrow \lambda = 0 \end{array} \right)$$

Proposition: Let $S \subset V$ be a subset of a vector space V. Then, the following are equivalent.

(1) S is linearly independent

(2) Each $\vec{x} \in \operatorname{span}(S)$ can be expressed in a unique way as a linear combination of vectors of S.

(3) The only representations of $\vec{0}$ as linear combinations of vectors of S are trivial representations, i.e., if

$$\vec{0} = a_1\vec{u}_1 + \dots + a_n\vec{u}_n \quad \text{for}$$

some $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n \in S$, $a_1, a_2, \dots, a_n \in F$, then we must have $a_1 = a_2 = \dots = a_n = 0$

Example: · For $k=0,1,2,\dots,n$, let $f_k(x)=1+x+x^2+\dots+x^k$.

Then: $S=\{f_0(x), f_1(x), f_2(x), \dots, f_n(x)\}\subset P_n(F)$ is a linearly independent subset.

$$0=\vec{0}=a_0f_0(x)+a_1f_1(x)+\dots+a_nf_n(x)$$

$$=a_0+a_1(1+x)+a_2(1+x+x^2)+\dots+a_n(1+x+\dots+x^n)$$

$$=(a_0+a_1+\dots+a_n)1+(a_1+a_2+\dots+a_n)x$$
$$+(a_2+a_3+\dots+a_n)x^2+\dots+a_nx^n$$

$$\left\{\begin{aligned} a_0+a_1+\dots+a_n&=0\\ a_1+\dots+a_n&=0\\ a_2+\dots+a_n&=0\\ a_n&=0\end{aligned}\right. \Rightarrow a_1=a_2=\dots=a_n=0.$$

Theorem: Let S be a linearly independent subset of a vector space V. Let $\vec{v} \in V \setminus S$. Then: $S \cup \{\vec{v}\}$ is linearly dependent iff $\vec{v} \in \mathrm{Span}(S)$.

Definition: A basis for a vector space V is a subset $\beta \subset V$ such that:

- β is linearly independent and
- β spans V, i.e. $\mathrm{Span}(\beta) = V$.

e.g. • F^n : $\{ \vec{e}_1 = (1, 0, \ldots, 0), \vec{e}_2 = (0, 1, 0 \ldots, 0), \ldots, \vec{e}_i = (0, .., 0, \overset{\text{i-th}}{\overset{\downarrow}{1}}, 0 .. 0), \ldots, \vec{e}_n = (0, 0, \ldots, 1) \}$

is a basis for F^n. (Standard basis)

- $M_{2\times 2}(F)$: $\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & -2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} \right\} \subset M_{2\times 2}(F)$

  is a basis for $M_{2\times 2}(F)$

- $\{1, x, x^2, \ldots, x^n\}$ is a basis for

- $\{1, x, x^2, ------\}$ is a basis for

__Theorem:__ Let V be a vector space and $\beta = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\} \subset V$.

Then: β is basis for V if and only if: $\underset{\text{(for all)}}{\forall} \vec{v} \underset{\text{(in)}}{\in} V$, $\underset{\text{(there exist)}}{\exists} \overset{\text{(unique)}}{!}$ $a_1, a_2, \ldots, a_n \in F$ such that:

$$\vec{v} = a_1 \vec{u}_1 + a_2 \vec{u}_2 + \ldots + a_n \vec{u}_n.$$

Lemma: Let S be a linearly dependent subset of a vector space V.
Then: $\exists \vec{v} \in S$ such that $\operatorname{span}(S \setminus \{\vec{v}\}) = \operatorname{span}(S)$.

Theorem: Suppose S is a finite spanning set for a vector space V.
Then: $\exists \beta \subset S$ which is a basis for V.

(A finite spanning set can be reduced to a basis)

Theorem: Let V be a vector space.

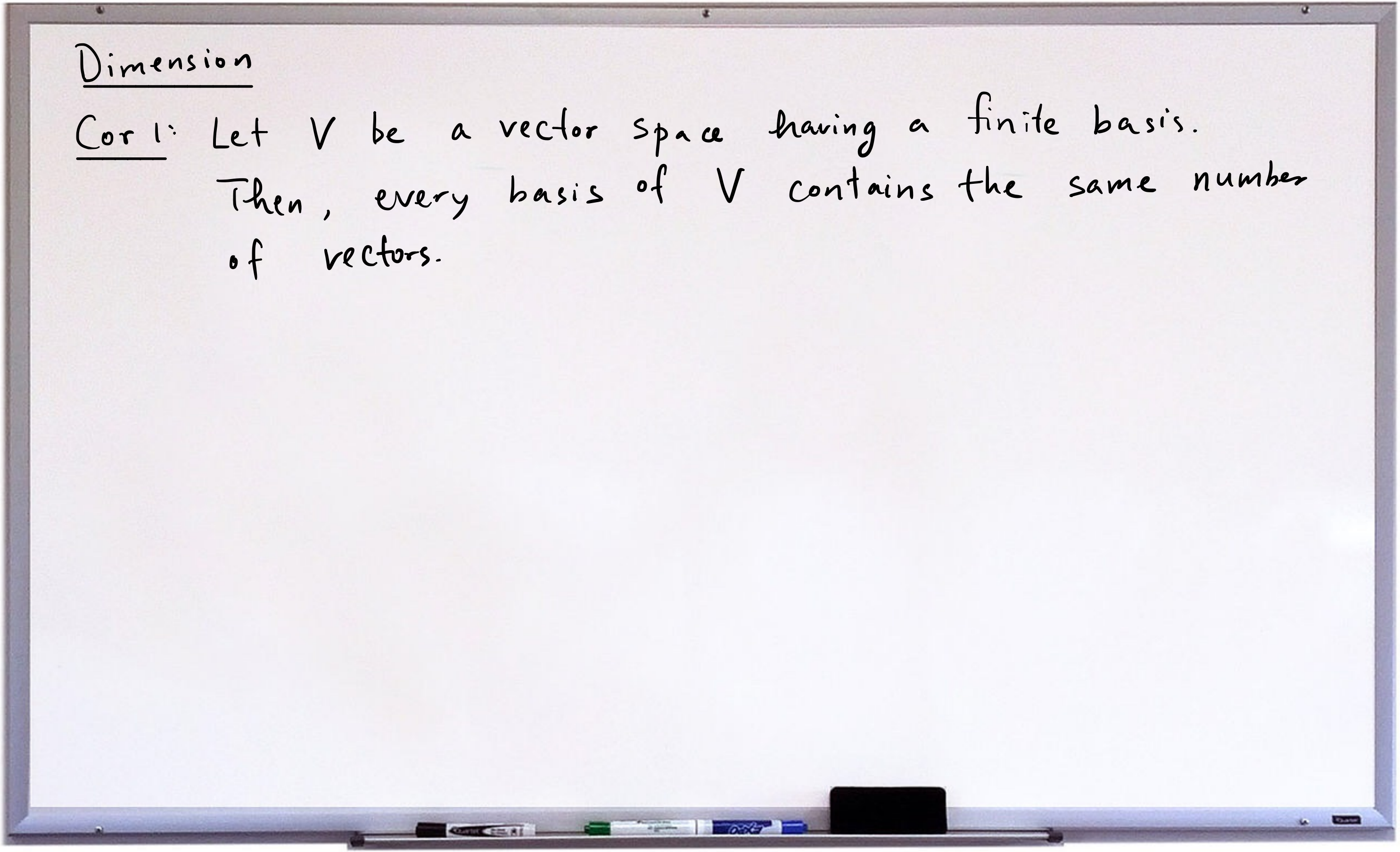
Let $G \subset V$ be a spanning set for V consisting of n vectors.

and $L \subset V$ be a linearly independent Subset consisting of m vectors.

Then, $m \leq n$ and $\exists\, H \subset G$ consisting of exactly $n-m$ vectors

such that $L \cup H$ spans V.

(Replacement thm)



Dimension

Cor 1: Let V be a vector space having a finite basis. Then, every basis of V contains the same number of vectors.

Definition: A vector space V is called **finite-dimensional** if it has a finite basis. The **dimension** of V, denoted as $\dim(V)$, is the number of vectors in a basis for V.

A vector space which is not finite-di al is called **infinite-dimensional**

Example:

- F^n is n-dimensional
- $\mathcal{F}(\mathbb{R}, \mathbb{R})$ is infinite-dimensional