Exercise

Note: you may take the follow results for granted in the following questions.
For any real a, b:

(a) $|a| = \sqrt{a^2}$

(b) $|a| \geq 0$

(c) $|a| = |-a|$

(d) $||a|| = |a|$

(e) $|ab| = |a||b|$

(f) $|a| = 0$ if and only if $a = 0$. In particular, $|a - b| = 0$ if and only if $a = b$

1. Solve the following equations and inequalities by sketching suitable graphs.

(a) $|(x+1)^2 - 10| = 6$

(b) $|x+1| + |x+4| = 5$

(c) $x < |x+2|$

(d) $|x-1| = |x| - 1$

2. For any real numbers x, y, determine whether the following are true or false:

(a) $\sqrt{x^2} = |x|$

(b) $(\sqrt{x})^2 = |x|$

(c) if $|x| = |y|$ then $x = y$

(d) if $x = y$ then $|x| = |y|$

(e) if $|x| = |y|$ then $x^2 = y^2$

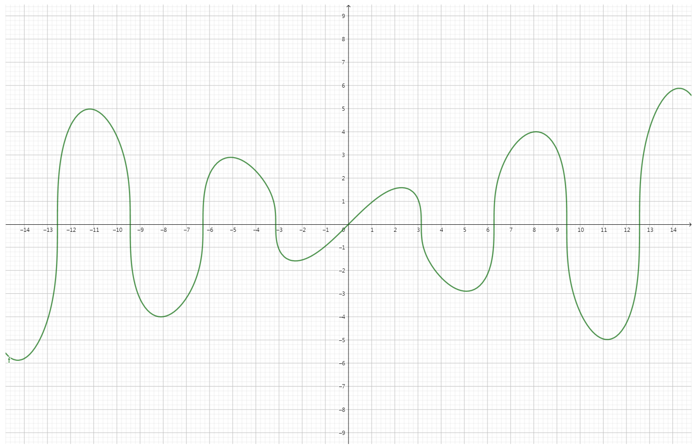
(f) if $x^2 = y^2$ then $|x| = |y|$

(g) $\frac{1}{|x|}$ is a real number

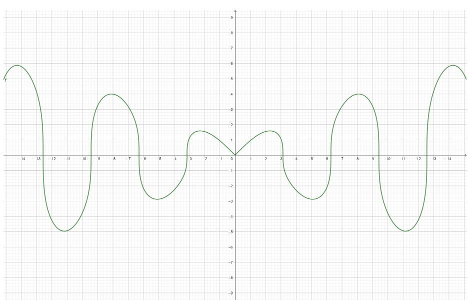
(h) $\sqrt{|x|} = |\sqrt{x}|$

(i) if $x \neq 0$ then $\log(x^2) = 2\log|x|$

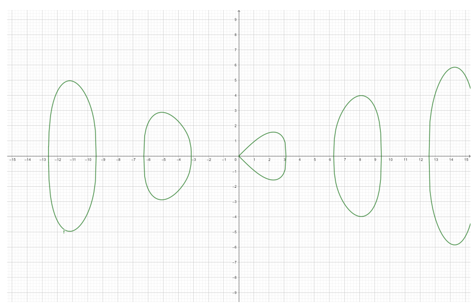
3. Identify the following graphs with the corresponding equations:



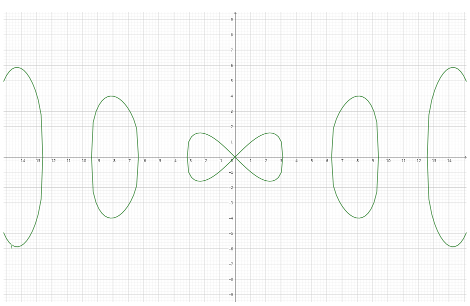
Graph of $y = f(x)$



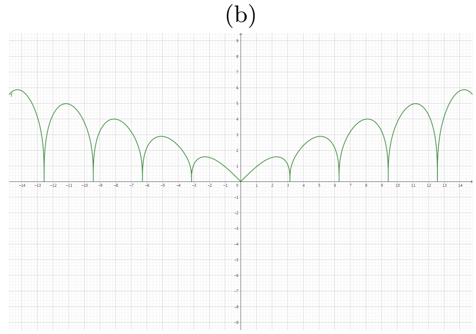
(a)



(b)



(c)



(d)

(I) $|y| = f(|x|)$ (II) $y = f(|x|)$ (III) $y = |f(x)|$ (IV) $|y| = f(x)$

4. Solve the following equations for real x:

(a) $|x^2+6x+9| = 2x-4$

(b) $|\frac{6}{x}-1| = 2$

(c) $(x+1)^2+6|x+1|-16=0$

(d) $|x^2-3x-10| = 6x$

(e) $|x^2-3x-4| = |x^2-1|$

(f) $|x^4-2x^2+5| = x^2+9$

(g) $|x-4|+|3x-1| = 9$

(h) $||x+1|-|2x+3|| = 3x$

(i) $||x^2-5x+6|+|x^2-4|| = x-2$

(j) $|x^2+8x+12| = x+6$

(k) $2(x-2)^2-3 = |4x-5|$

(l) $2|x+1|-3 = \frac{2}{|x+1|}$

(m) $|x+\frac{1}{x}|^2-2|x+\frac{1}{x}|-8=0$

(n) $|x^2+6x+5| = 2|x^2+6x+6|$

(o) $|x^2+2x-3| = |x^2-4x+3|$

(p) $|x+2|+|2x-7| = 13$

(q) $|5x-2|-|3x-4| = 10$

(r) $4e^x = |e^{2x}+3|$

(s) $|4e^x-5| = 3$

(t) $\log_2|x|+\log_2|3x+1| = 1$

(u) $\log(x^2)=\log(|x|)+2$

(v) $|\frac{3}{\log_3 x+7}| = 2$

(w) $\sin(|x|)=\cos(|x|)$, $-\pi \leq x < \pi$

(x) $|\sin(x)| = |\cos(x)|$, $-\pi \leq x < \pi$

Note: The following questions are more theoretical.

5. Prove the following properties about absolute value stated in the beginning. Here a, b are any real numbers:

(a) $|a| \geq 0$

(b) $|a| = \sqrt{a^2}$

(c) $|a| = |-a|$

(d) $||a|| = |a|$

(e) $|ab| = |a||b|$

(f) $|a| = 0$ if and only if $a = 0$. In particular, $|a - b| = 0$ if and only if $a = b$

6. For any real a, b, c, show that

(a) If $c \geq 0$, then $|a| \leq c$ if and only if $-c \leq a \leq c$ (the inequality still holds if $\leq$ is replaced by $<$.)

(b) $-|a| \leq a \leq |a|$

(c) $|a + b| \leq |a| + |b|$

(d) $|a - b| \geq ||a| - |b||$ (Tips: Try $a = x - y$, $b = y$ and $a = y - x$, $b = x$ on 6.(c).)

(e) $|a - b| \leq |a - c| + |c - b|$

Remark: The result in 6.(c) is known as Triangle Inequality (in reals) and that in 6.(d) is known as Reversed Triangle Inequality. Details will be discussed in the topic "Inequality".

7. We define $\max(\{a, b\})$ and $\min(\{a, b\})$ as the greater and smaller value between a and b respectively. For example, $\max(\{-2, 4\}) = 4$ and $\min(\{1, 10\}) = 1$.

Show that $\max(\{a, b\}) = \frac{a+b+|a-b|}{2}$ and $\min(\{a, b\}) = \frac{a+b-|a-b|}{2}$.

Note: Then we see that $|a - b| = \max(\{a, b\}) - \min(\{a, b\})$, which means the difference between a and b on real number line.

8. Prove the following results:

(a) $|a + b + c| \leq |a| + |b| + |c| \quad \forall a, b, c \in \mathbb{R}$.

(b) $|a + b| \leq 2\sqrt{a^2 + b^2} \quad \forall a, b \in \mathbb{R}$. (Hint: apply 5.(a) and 6.(c))

(c) Let $a, b \in \mathbb{R}$ and $c > 0$. If a, b are solutions to $|x + 1| < c$, then $|a - b| < 2c$. (Hint: apply 6.(e))

(d) Let $a > 0$. $\forall x \in \mathbb{R}$, if $|x - 2| < a$ then $|x^2 - 4| < (a + 2)^2$. (Hint: apply 6.(a) and 5.(e))

(e) the equation $|x + 1| = |x| + 2$ has no real solution. (Hint: apply 6.(c))

(f) the solution to the equation $||x + 2| - |x - 3|| < 6$ is all real numbers. (Hint: apply 6.(d))