Exercise

1.(a) $x < \frac{1}{2}$ or $\frac{3}{2} < x < 3$

1.(b) $x < -1$ or $x > 2$

1.(c) $x = -1$ or $x \geq 1$

1.(d) $x < 3$ or $3 < x < 10$

1.(e) $x < 6$ or $x > \frac{20}{3}$

1.(f) $-4 \leq x \leq -2$

1.(g) $\frac{1}{2} < x < 2$

1.(h) $x < -5$ or $0 < x < 2$

1.(i) $2n\pi + \frac{\pi}{4} < x < 2n\pi + \frac{\pi}{2}$
or $2n\pi + \frac{3\pi}{2} < x < 2n\pi + \frac{7\pi}{4}$, $n \in \mathbb{Z}$

1.(j) $0 < x < \frac{1}{27}$ or $x > 9$

1.(k) $1 < x \leq 9$

1.(l) $2n\pi - \frac{3\pi}{4} < x < 2n\pi + \frac{\pi}{4}$, $n \in \mathbb{Z}$

1.(m) $\log(n\pi - \frac{\pi}{2}) < x < \log(n\pi)$, $n \in \mathbb{Z}, n \geq 1$

1.(n) $2n\pi - \frac{3\pi}{4} \leq x < 2n\pi - \frac{\pi}{2}$, $n \in \mathbb{Z}$

1.(o) $2n\pi < x < 2n\pi + \pi$, $n \in \mathbb{Z}, n \leq -1$
or $2n\pi - \pi < x < 2n\pi$, $n \in \mathbb{Z}, n \geq 1$

1.(p) $\frac{\pi}{2} \leq x \leq \frac{11\pi}{6}$

1.(q) $\frac{\pi}{4} < x < \frac{3\pi}{4}$ or $\pi < x < \frac{5\pi}{4}$
or $\frac{7\pi}{4} < x < 2\pi$

1.(r) $x < -3$ or $3 < x < 4$

1.(s) No real solution

1.(t) $1 < x \leq 3$

1.(u) $-4 \leq x < -3$ or $-1 < x \leq 4$

1.(v) $x < 0$ or $x > 1$

1.(w) $0 < x < 1$ or $x > 4$

2.(a) false
2.(b) true
2.(c) false
2.(d) true
2.(e) false
2.(f) false
2.(g) true
2.(h) false
2.(i) true
2.(j) true
2.(k) false
2.(l) false
2.(m) false
2.(n) false
2.(o) true

3.(a) $1 \leq x^2 - 4x + 5 \leq 37$

3.(b) $\frac{1}{3^{12}} \leq (\frac{1}{3})^{x^2+5x+6} \leq \frac{1}{9}$

3.(c) $1 \leq \log_2(\sin(x) + 3) \leq 2$

3.(d) $-\frac{2}{7} \leq \frac{x^2+6x+4}{x^2-3x+4} \leq 10$

3.(e) $\frac{(x+3)^2}{x-1} \geq 16$

3.(f) $-1 \leq \frac{\sin(2x)}{1+\cos(2x)} \leq 1$

3.(g) $-1 \leq \frac{\cos(x)-\sin(x)}{\cos(x)+\sin(x)} \leq 2 - \sqrt{3}$

3.(h) $\frac{1}{4} \leq \frac{4-\sin(x)}{\cos^2(x)+\sin(x)+11} \leq \frac{1}{2}$

3.(i) $0 \leq \log_{\frac{1}{x}}(\frac{x}{4}) \leq 1$

Selected examples:

3.(d) Assume $x = 0$. Then $\frac{x^2+6x+4}{x^2-3x+4} = 1$. If $x \neq 0$, then $\frac{x+6+\frac{4}{x}}{x-3+\frac{4}{x}} = 1 + \frac{9}{x-3+\frac{4}{x}}$
(Case 1) Assume $x > 0$. Then $x - 3 + \frac{4}{x} = (\sqrt{x} - \frac{2}{\sqrt{x}})^2 + 1 \geq 1$.
Hence $1 < 1 + \frac{9}{x-3+\frac{4}{x}} \leq 10$.
(Case 2) Assume $x < 0$. Then $x - 3 + \frac{4}{x} = -(\sqrt{-x} - \frac{2}{\sqrt{-x}})^2 - 7 \leq -7$.
Hence $-\frac{2}{7} \leq 1 + \frac{9}{x-3+\frac{4}{x}} < 1$.
Combining all these cases together, we have $-\frac{2}{7} \leq \frac{x^2+6x+4}{x^2-3x+4} \leq 10$.

3.(d) (Alternative approach) Notice that $x^2 - 3x + 4 \neq 0$ for all x.
Let $y = \frac{x^2+6x+4}{x^2-3x+4}$. Then $(y-1)x^2 + (-3y-6)x + (4y-4) = 0$.
Now, there must be real root to this equation. So, $\Delta = (3y+6)^2 - 4(y-1)(4y-4) \geq 0$.
Solving gives $-\frac{2}{7} \leq y \leq 10$.

3.(e) $\frac{(x+3)^2}{x-1} = \frac{x^2+6x+9}{x-1} = x + 7 + \frac{16}{x-1} = x - 1 + 8 + \frac{16}{x-1}$
Since $x > 1$, $x - 1 + 8 + \frac{16}{x-1} = (\sqrt{x-1} - \frac{4}{\sqrt{x-1}})^2 + 16 \geq 16$.
Note that $\frac{(x+3)^2}{x-1}$ is unbounded above for $x > 1$.

4.(e) Note that $\sqrt{x-1} < \sqrt{x} < \sqrt{x+1}$. So $0 < \sqrt{x-1} + \sqrt{x} < 2\sqrt{x} < \sqrt{x} + \sqrt{x+1}$.
Then, $\frac{1}{\sqrt{x-1}+\sqrt{x}} > \frac{1}{2\sqrt{x}} > \frac{1}{\sqrt{x}+\sqrt{x+1}}$.
Now, $\frac{1}{\sqrt{x-1}+\sqrt{x}} = \sqrt{x} - \sqrt{x-1}$ and $\frac{1}{\sqrt{x}+\sqrt{x+1}} = \sqrt{x+1} - \sqrt{x}$.
Hence we have $2\sqrt{x+1} - 2\sqrt{x} < \frac{1}{\sqrt{x}} < 2\sqrt{x} - 2\sqrt{x-1}$.

4.(f) Note that $2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca = (a-b)^2 + (b-c)^2 + (c-a)^2 \geq 0$.
So $a^2 + b^2 + c^2 \geq ab + bc + ca$.
Hence $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca \leq 3(a^2 + b^2 + c^2)$.
Also, $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca \geq 3(ab + bc + ca)$.
Equality holds iff $a = b = c$.

4.(j) From 4.(i), we have $\frac{a+b+c+d}{4} \geq \sqrt[4]{abcd}$ for $a, b, c, d \geq 0$.
Now consider $a = u, b = v, c = w, d = \frac{u+v+w}{3}$. We have $\frac{a+b+c+d}{4} = \frac{u+v+w+\frac{u+v+w}{3}}{4} = \frac{u+v+w}{3} = d$.
Note that $d = 0$ iff $u, v, w = 0$, in which case the inequality is obvious. So assume $d > 0$.
Since $d = \frac{a+b+c+d}{4} \geq \sqrt[4]{abcd}$, $\sqrt[4]{d^3} \geq \sqrt[4]{abc}$.
So $d^3 \geq abc$ and hence $\frac{u+v+w}{3} = d \geq \sqrt[3]{abc} = \sqrt[3]{uvw}$.

4.(l) Since $a, b, c > 0$, $(a + \frac{1}{b}), (b + \frac{1}{c}), (c + \frac{1}{a}) > 0$.
Now, $(a + \frac{1}{b}) \geq 2\sqrt{\frac{a}{b}}$, $(b + \frac{1}{c}) \geq 2\sqrt{\frac{b}{c}}$, $(c + \frac{1}{a}) \geq 2\sqrt{\frac{c}{a}}$.
So $(a + \frac{1}{b})(b + \frac{1}{c})(c + \frac{1}{a}) \geq (2\sqrt{\frac{a}{b}})(2\sqrt{\frac{b}{c}})(2\sqrt{\frac{c}{a}}) = 8$.
Equality holds iff $a = b = c$.

5.(a) Negative

5.(b) Suppose $a_2 < m < a_3$.
Then $c, m - a_1, m - a_2 < 0$ and $m - a_3, ..., m - a_8 > 0$.
Hence $f(m) = c(m - a_1)(m - a_2)...(m - a_8) < 0$.

5.(c) $a_n \leq x \leq a_{n+1}$, $n = 1, 3, 5, 7$

5.(d)

6.(b) Local maximum at $x = \frac{3\pi}{2}$, $y = \text{csc}(\frac{3\pi}{4}) = -1$
Local minimum at $x = \frac{\pi}{2}$, $y = \text{csc}(\frac{\pi}{2}) = 1$

6.(c) Take $\delta = \frac{1}{2}$.
Notice that when $-\delta < x < \delta$, $0 \leq x^2 < \delta^2 = \frac{1}{4}$.
So $x^4 - x^2 = x^2(x^2 - 1) \leq 0$.

6.(e) Assume $f(x) = \frac{1}{x}$ attains maximum at $a > 0$.
So there is $\delta > 0$ such that if $0 < a - \delta < x < a + \delta$ then $f(x) \leq f(a)$.
However $f(x)$ is strictly decreasing for $x > 0$, so $f(a - \frac{\delta}{2}) > f(a)$.
Contradiction arises. Proof for local minimum is similar.

7.(b) 4n-2

7.(d) $\beta - \alpha = \sqrt{(\beta + \alpha)^2 - 4\alpha\beta} = \sqrt{(2 - n)^2 - 4(n - 5)} = \sqrt{n^2 - 8n + 24} < \sqrt{44}$
So $n^2 - 8n + 24 < 44$ and hence $-2 < n < 10$.
Combined with 7.(c) gives $-2 < n < \frac{1}{2}$.

8.(a) Suppose $x > y$. Then $e^x > e^y$ and $e^{-x} < e^{-y}$.
So $\frac{e^x - e^{-x}}{2} > \frac{e^y - e^{-y}}{2}$.

8.(b)

Note that $\dfrac{e^x + e^{-x}}{2} = \dfrac{(e^{\frac{x}{2}} - e^{-\frac{x}{2}})^2}{2} + 1$
Also, $e^{\frac{x}{2}} - e^{-\frac{x}{2}}$ is strictly increasing and passes through (0,0).

8.(d) $\frac{e^x - e^{-x}}{e^x + e^{-x}} = 1 - \frac{2e^{-x}}{e^x + e^{-x}} = 1 - \frac{2}{e^{2x}+1}$

9.(b) Since $\sqrt{x} \neq \frac{1}{\sqrt{x}}$, $(\sqrt{x} - \frac{1}{\sqrt{x}})^2 > 0$.
So $y = x + \frac{1}{x} = (\sqrt{x} - \frac{1}{\sqrt{x}})^2 + 2 > 2$.
Then, $y^2 > 2y > y + 2$. Hence $y^2 - y - 2 > 0$.