# Exercise

1. Solve the following inequalities, using appropriate methods as needed:

   (a) $(2x-1)(x-3)(3-2x) > 0$

   (b) $(x+1)(x-2)(x^2+2x+4) > 0$

   (c) $x^3+x^2-x-1 \geq 0$

   (d) $(x-3)(x^2-13x+13) < 51-17x$

   (e) $\frac{4}{x-6} < 6$

   (f) $\frac{6}{x^2+6x+10} \geq 3$

   (g) $\frac{x+1}{2-x} > 1$

   (h) $x+3 < \frac{10}{x}$

   (i) $\sec(x) > \sqrt{2}$

   (j) $(\log_3(x))^2 + 2\log_9(3x) > 7$

   (k) $\log_x(4) \geq \log_3(2)$

   (l) $5^{\sin(x)} < 5^{\cos(x)}$

   (m) $\tan(e^x) < 0$

   (n) $2\sin(x) \leq 1 \leq \tan(x)$

   (o) $x\sin(x) < 0$

   (p) $\sin(x) + \sqrt{3}\cos(x) \leq 1$ and $0 \leq x < 2\pi$ (Hint: Assume $\sin(x) + \sqrt{3}\cos(x) = A\cos(x+B)$)

   (q) $(x-\pi)\cos(2x)e^x > 0$ and $0 \leq x < 2\pi$

   (r) $3x-4 < 8$ and $x^2-8 > 1$

   (s) $\frac{2}{x+3} \geq 1$ and $\ln(x) > 6$

   (t) $\log_2(x) > \log_3(x)$ and $x^2-2x-3 \leq 0$

   (u) $(2x+6 < 0$ or $3x+5 > 2)$ and $x^2 \leq 16$

   (v) $(xe^x < 0$ and $-6 \leq 2x-2 \leq -4)$ or $\frac{1}{x} < 1$

   (w) $(\sin(2^x) > 0$ and $\log_2(5) < x < \log_2(6))$ or $(\log_x(4) < 1$ and $x > 0)$

2. Determine whether each of the following statement is true or false:

(a) For any real number $x$, $\frac{1}{x^2} \geq 0$.

(b) The inequality $\cos(x) > 1$ has no real solution.

(c) If $x > y$ then $x^2 > y^2$.

(d) If $c < 0$ and $x > y$ then $cx < cy$.

(e) $\sec(x) \geq \cos(x)$ for all real number $x$.

(f) If $\sin(x) < \sin(y)$ then $x < y$.

(g) $x^3 + \frac{1}{x^3} \geq x + \frac{1}{x}$ for all $x > 0$.

(h) If $a, b, x > 0$ and $\log_a(x) > \log_b(x)$ then $a < b$.

(i) If $\log_x(2) < 5$ then $x > \sqrt[5]{2}$ or $0 < x < 1$.

(j) $2020^{2021} > 2021^{2020}$.

(k) The maximum value for $\sin(x)+\cos(2x)$ is 2.

(l) An increasing function cannot be decreasing.

(m) If $f(x)$ is strictly decreasing, then there is one and only one solution to $f(x) = 0$.

(n) If both $f(x), g(x)$ are strictly increasing then $f(x)g(x)$ is strictly increasing.

(o) If both $f(x), g(x)$ are strictly decreasing then $f(g(x))$ is strictly increasing.

3. Find the range of the following expressions:

(a) $x^2 - 4x + 5$, $-4 \leq x \leq 4$

(b) $(\frac{1}{3})^{x^2+5x+6}$, $-1 \leq x \leq 1$

(c) $\log_2(\sin(x) + 3)$

(d) $\frac{x^2+6x+4}{x^2-3x+4}$ (Hint: Divide both numerator and denominator by $x$)

(e) $\frac{(x+3)^2}{x-1}$, $x > 1$ (Hint: Transform the expression into $A(x-1) + B + \frac{C}{x-1}$)

(f) $\frac{\sin(2x)}{1+\cos(2x)}$, $-\frac{\pi}{4} \leq x \leq \frac{\pi}{4}$

(g) $\frac{\cos(x)-\sin(x)}{\cos(x)+\sin(x)}$, $0 \leq x \leq \frac{\pi}{3}$

(h) $\frac{4-\sin(x)}{\cos^2(x)+\sin(x)+11}$

(i) $\log_{\frac{1}{x}}(\frac{x}{4})$, $2 \le x \le 4$

4. Prove the following inequalities (Variables are real unless otherwise specified):

(a) $|\sin(x)| \le |\tan(x)|$, $x \ne n\pi + \frac{\pi}{2}, n \in \mathbb{Z}$

(b) $\log_2(3) < \log_3(5)$

(c) $e^{x^2+5x+6} > \sin(x)$

(d) $2\sqrt{x+1} - 2\sqrt{x} < \frac{1}{\sqrt{x}} < 2\sqrt{x} - 2\sqrt{x-1}$, $x > 1$ (Hint: Take reciprocal)

(e) $\frac{a}{b} + \frac{b}{a} \ge 2$, where $a, b > 0$

(f) $3(a^2+b^2+c^2) \ge (a+b+c)^2 \ge 3(ab+bc+ca)$ (Hint: Show $a^2+b^2+c^2 \ge ab+bc+ca$)

(g) $a^3+b^3+c^3+d^3 \le 27$, where $a^2+b^2+c^2+d^2 = 9$ (Hint: Show $a^3 \le 3a^2$)

(h) $a^2+4b^4+c^2+1 \ge b(4ab-b-2c) - a(ac^2+c)$

(i) $\frac{a+b+c+d}{4} \ge \sqrt[4]{abcd}$, where $a, b, c, d \ge 0$ (Hint: $\sqrt[4]{abcd} = \sqrt{\sqrt{ab}\sqrt{cd}}$. Let $u = \sqrt{ab}, v = \sqrt{cd}$. )

(j) $\frac{u+v+w}{3} \ge \sqrt[3]{uvw}$, where $u, v, w \ge 0$ (Hint: Use previous result with $a = u, b = v, c = w$, $d = \frac{u+v+w}{3}$. So $d = \frac{a+b+c+d}{4}$. Now show $d \ge \sqrt[3]{abc}$ )

(k) $(ab^2+bc^2+ca^2)(a^2b+b^2c+c^2a) \ge 9(abc)^2$, where $a, b, c \ge 0$ (Hint: Use previous result)

(l) $(a+\frac{1}{b})(b+\frac{1}{c})(c+\frac{1}{a}) \ge 8$, where $a, b, c > 0$ (Hint: Use $\frac{a+b}{2} \ge \sqrt{ab}$)

5. Let $f(x) = c(x-a_1)(x-a_2)...(x-a_8)$, where $a_1, a_2, ..., a_8$ are real numbers with $a_1 < a_2 < ... < a_8$ and $c \ne 0$. Suppose $a_5 < r < a_6$ and $f(r) > 0$.

(a) Is $c$ positive or negative?

(b) Show that if $a_2 < m < a_3$ then $f(m) < 0$.

(c) Solve $f(x) \ge 0$.

(d) Sketch the graph of $y = f(x)$.

6. In this question, we will talk about local maximum and minimum:

Let $f(x)$ be a function. $f(x)$ is said to attain local maximum at $x = a$ if there exists $\delta > 0$ such that $f(x) \leq f(a)$ whenever $a - \delta < x < a + \delta$.

Similarly, $f(x)$ is said to attain local minimum at $x = a$ if there exists $\delta > 0$ such that $f(x) \geq f(a)$ whenever $a - \delta < x < a + \delta$.

Conceptually, it means you can find a small neighbourhood near $a$ such that $f(a)$ is larger than (or smaller than) all $f(x)$ within that neighbourhood.

(a) Find an example such that the function attains local minimum but not global minimum.

(b) Identify the local maximum and local minimum of $y = \csc(x)$ for $0 < x < 2\pi$.

(c) Show that $f(x) = x^4 - x^2$ attains local maximum at $x = 0$.

(d) Show that $f(x) = |\ln(x)|$ attains local minimum at $x = 1$.

(e) Show that $f(x) = \frac{1}{x}$ attains neither local maximum nor local minimum for $x > 0$.

(f) Show that if $f(x)$ does not attain local maximum, then it does not attain global maximum.

(g) Show that if $f(x)$ attains local minimum at $x = a$, then there exists $\delta > 0$ such that $f(x)$ is decreasing on $(a - \delta, a]$ and increasing on $[a, a + \delta)$.

7. Let $f(x) = x^2 + (n-2)x + (n-5)$, where $n$ is a real number. Suppose $\alpha, \beta$ are the roots of $f(x)$.

(a) Show that $\alpha$ and $\beta$ are real and $\alpha \neq \beta$.

(b) Express $(\alpha - 3)(\beta - 3)$ in terms of $n$.

(c) Assume $\alpha < 3 < \beta$. Using the previous result, show that $n < \frac{1}{2}$.

(d) Further suppose $\beta - \alpha < \sqrt{44}$. Show that $-2 < n < \frac{1}{2}$.

8. In this question, we will introduce hyperbolic functions. Define:

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

These functions are useful in physics calculations.

(a) Show that $\sinh(x)$ is strictly increasing.

(b) Show that $\cosh(x)$ is strictly decreasing on $(-\infty, 0]$ and strictly increasing on $[0, \infty)$.

(c) Hence, or otherwise, show that $\cosh(x) \geq 1$ for all $x$.

(d) Show that $\tanh(x)$ is strictly increasing. (Hint: Reduce amount of variables)

(e) Show that $|\sinh(x)| < |\cosh(x)|$ for all $x$.

(f) Hence, or otherwise, show that $-1 < \tanh(x) < 1$ for all $x$.

9. In this question, we will introduce another proof to the inequality "$x^2 + \frac{1}{x^2} > x + \frac{1}{x}$ for any $1 \neq x > 0$". Let $1 \neq x > 0$, and $y = x + \frac{1}{x}$

(a) By splitting cases for $0 < x < 1$ and $x > 1$, prove that $\sqrt{x} \neq \frac{1}{\sqrt{x}}$.

(b) By (a), show that $y > 2$. Hence, show that $y^2 - y - 2 > 0$.

(c) Hence, show that $x^2 + \frac{1}{x^2} > x + \frac{1}{x}$.