Binomial Theorem Exercise (part 2)

June 2, 2021

1 Elementary questions

Q1) Evaluate

$$\binom{9}{0}+\binom{9}{1}+\binom{9}{2}+\binom{9}{3}+...+\binom{9}{8}+\binom{9}{9}$$

Q2) Evaluate

$$\binom{4n}{0}+\binom{4n}{2}+\binom{4n}{4}+\binom{4n}{6}+...+\binom{4n}{4n-2}+\binom{4n}{4n}$$

and express your answer in terms of n
Q3) Evaluate

$$\binom{20}{1}-\binom{20}{3}+\binom{20}{5}-\binom{20}{7}+...+\binom{20}{17}-\binom{20}{19}$$

Q4) Evaluate

$$\binom{4n+1}{1}-\binom{4n+1}{3}+\binom{4n+1}{5}-\binom{4n+1}{7}+...-\binom{4n+1}{4n-1}+\binom{4n+1}{4n+1}$$

and express your answer in terms of n

2 Intermediate level questions

Q5a) Show that,for any natural number n,r with $n \geq r$, we have

$$r \times \binom{n}{r} = n \times \binom{n-1}{r-1}$$

Q5b) By part a), or otherwise, show that

$$\sum_{k=0}^{n} r \times \binom{n}{r} = n \times 2^{n-1}$$

Q5c) Can you also evaluate the sum

$$\sum_{k=0}^{n} r \times (r-1) \times \binom{n}{r}$$

Q6a) Show that,for any natural number n,r with $n \geq r$, we have

$$\binom{n}{r} \times \frac{1}{r+1} = \binom{n+1}{r+1} \times \frac{1}{n+1}$$

Q6b) Hence, or otherwise, show that

$$\sum_{k=0}^{n} \binom{n}{r} \times \frac{1}{r+1} = \frac{2^{n+1}}{n+1}$$

Q7) By considering $(1+x)^{2n} = (1+x)^n \times (1+x)^n$, show that

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + ... + \binom{n}{n-1}^2 + \binom{n}{n}^2 = \frac{(2n)!}{(n!)^2}$$

Q8) By considering $(1+x)^{2n} = (1+x)^n \times (1+x)^n$ and the coefficient of $x^{(n+2)}$,show that

$$\sum_{k=0}^{n-2} \binom{n}{k} \times \binom{n}{k+2} = \frac{(2n)!}{(n-2)! \times (n+2)!}$$

Q9) In the lesson, we have learnt to find the sum

$$\sum_{r=0}^{n} \binom{2n}{2r} = 2^{(2n-1)}$$

. We now learn another way to evaluate the sum.
Q9a) Expand$(1+x^2)^{2n}$, by Binomial theorem.
Q9b) by putting $x = 1$ and $x = i$ in the expansion in (a), show that

$$\sum_{r=0}^{2n} \binom{2n}{r} = 2^{2n}$$

and

$$\sum_{r=0}^{2n} (-1)^r \binom{2n}{r} = 0$$

Q9c) By using part b), evaluate the sum

$$\sum_{r=0}^{n} \binom{2n}{2r}$$

and express your answer in terms of n

3 Challenging questions

Q10a) Let, m and n be positive integers, express the sum$\sum_{r=0}^{m} (1+x)^{n+r}$ in the form of

$$\frac{(1+x)^{A+B+1}-(1+x)^{C}}{x}$$

where A,B,C are some numbers (may express it in terms of m,n)
Q10b) By part a), show that

$$\binom{n}{n}+\binom{n+1}{n}+\binom{n+2}{n}+\binom{n+3}{n}+...+\binom{n+m-1}{n}+\binom{n+m}{n} = \binom{n+m+1}{n+1}$$

Q10c) By part b), show that

$$\sum_{k=4}^{n+3} k(k-1)(k-2) = 6 \times (\binom{n+4}{4} - 1)$$

Hence evaluate the sum

$$\sum_{k=0}^{m} k(k-1)(k-2)$$

for $m \geq 3$
Q10d) Can you also evaluate the sum

$$\sum_{k=0}^{m} k(k-1)(k-2)(k-3)$$

and

$$\sum_{k=0}^{m} k(k-1)(k-2)(k-3)(k-4)$$

for $m \geq 4$ and $m \geq 5$ respectively?
(Question modified from HKAL 1994)

Q11a) Find all the roots of the equation $1 + x + x^2 + x^3 = 0$

Q11b) By using part a and considering $(1 + x + x^2 + x^3)^{4n}$, or otherwise, find the sum of the following expression(x_r represent the coefficient of x^r of $(1 + x + x^2 + x^3)^{4n}$):

bi)

$$\sum_{r=0}^{4n} (-1)^r x_r$$

bii)

$$\sum_{r=0}^{n} x_{4r}$$

biii)

$$\sum_{r=0}^{n-1} x_{4r+1}$$

biv)

$$\sum_{r=0}^{n-1} x_{4r+2}$$

bv)

$$\sum_{r=0}^{n-1} x_{4r+3}$$

Q12) By considering $(1 + x + x^2)^{3n}$, show that:

a)

$$\sum_{r=0}^{n} x_{3r} = 3^{3n-1}$$

b)

$$\sum_{r=0}^{n-1} x_{3r+1} = 3^{3n-1}$$

c)

$$\sum_{r=0}^{n-1} x_{3r+2} = 3^{3n-1}$$

where x_r represent the coefficient of x^r of $(1 + x + x^2)^{3n}$