MMAT5270 Introduction to Inverse Problems

Assignment 1

1 Problems:

3.2 Galerkin Discretization of Gravity Problem

As an example of the use of the Galerkin discretization method, discretize the gravity surveying problem from Section 2.1 with the "top hat" functions $\phi_j(t) = \chi_j(t)$ in (3.7) as the basis functions, and the functions $\psi(s) = \delta(s - s_i)$ with $s_i = (i - \frac{1}{2})h$ (corresponding to sampling the right-hand side at the s_i). Derive an expression for the matrix elements a_{ij} based on (3.5).

Solution.
First, we have the "top hat" function $\phi_j(t) = \chi_j(t) = \begin{cases} h^{-\frac{1}{2}}, & t \in [(j-1)h, jh] \\ 0, & \text{elsewhere.} \end{cases}$

Then, the matrix elements are given by:

$$
\begin{aligned}
a_{ij} &= \int_0^1 \int_0^1 \psi_i(s) K(s,t) \phi_j(t) ds\ dt \\
&= \int_0^1 \int_0^1 \delta(s - s_i) K(s,t) ds\ \phi_j(t) dt \\
&= \int_0^1 K(s_i, t) \phi_j(t) dt \\
&= h^{-\frac{1}{2}} \int_{(j-1)h}^{jh} K(s_i, t) dt
\end{aligned}
$$

□

3.3 Derivation of Important SVD Expressions

Give the details of the derivation of the important equations (3.9) and (3.10), as well as the expression (3.11) for the naive solution.

The solution x_{LS} to the linear least squares problem $\min_x \|Ax - b\|_2$ is formally given by $x_{LS} = (A^T A)^{-1} A^T b$, under the assumption that A has more rows than columns and $A^T A$ has full rank. Use this expression together with the SVD to show that x_{LS} has the same SVD expansion (3.11) as the naive solution.

Solution.
First, we prove equations (3.9):

$$Av_i = \sigma_i u_i, \qquad \|Av_i\|_2 = \sigma_i, \qquad i = 1, \ldots, n.$$

Proof:

$$
\begin{aligned}
Av_i &= \sum_{j=1}^{n} u_j \sigma_j v_j^T v_i \\
&= \sum_{j=1}^{n} u_j \sigma_j \delta_{ij} \\
&= u_i \sigma_i \delta_{ii} \\
&= \sigma_i u_i \\
\|Av_i\|_2 &= \|\sigma_i u_i\|_2 = \sigma_i \|u_i\|_2 = \sigma_i.
\end{aligned}
$$

Next, we prove equations (3.10):

$$A^{-1}u_i = \sigma_i^{-1}v_i, \qquad \|A^{-1}u_i\|_2 = \sigma_i^{-1}, \qquad i = 1, \dots, n.$$

Proof: By equations (3.9):

$$\begin{aligned} Av_i &= \sigma_i u_i \\ A^{-1}Av_i &= A^{-1}\sigma_i u_i \\ v_i &= \sigma_i A^{-1}u_i \\ \sigma_i^{-1}v_i &= A^{-1}u_i \\ \|A^{-1}u_i\|_2 &= \|\sigma_i^{-1}v_i\|_2 = \sigma_i^{-1}\|v_i\|_2 = \sigma_i^{-1}. \end{aligned}$$

Last, we prove equations (3.11):

$$x = A^{-1}b = \sum_{i=1}^{n} \frac{u_i^T b}{\sigma_i} v_i.$$

Proof: First note that since the matrix V is orthogonal, we can always write the vector x in the form

$$x = VV^T x = V \begin{pmatrix} v_1^T x \\ \vdots \\ v_n^T x \end{pmatrix} = \sum_{i=1}^{n} (v_i^T x) v_i,$$

and similarly for b we have

$$b = \sum_{i=1}^{n} (u_i^T b) u_i.$$

When we use the expression for x, together with the SVD, we obtain

$$\begin{aligned} Ax &= A \sum_{i=1}^{n} (v_i^T x) v_i \\ &= \sum_{i=1}^{n} (v_i^T x) A v_i \\ &= \sum_{i=1}^{n} (v_i^T x) \sigma_i u_i \\ &= \sum_{i=1}^{n} \sigma_i (v_i^T x) u_i \end{aligned}$$

By equating the expressions for A x and b, and comparing the coefficients in the expansions, we have

$$u_i^T b = \sigma_i (v_i^T x), \qquad i = 1, \dots, n$$

Hence, we get

$$\begin{aligned} x &= \sum_{i=1}^{n} (v_i^T x) v_i \\ &= \sum_{i=1}^{n} \frac{u_i^T b}{\sigma_i} v_i. \end{aligned}$$

For the case of linear least squares problem, we have

$$A^T A = (U\Sigma V^T)^T U\Sigma V^T = (V\Sigma U^T) U\Sigma V^T = V\Sigma^2 V^T = \sum_{i=1}^{n} v_i \sigma^2 v_i^T.$$

Then, we use the above expression for $A^T A$, we obtain

$$\begin{aligned} A^T A x_{LS} &= \sum_{i=1}^{n} v_i \sigma_i^2 v_i^T x_{LS} \\ &= \sum_{i=1}^{n} v_i \sigma_i^2 (v_i^T x_{LS}) \\ &= \sum_{i=1}^{n} \sigma_i^2 (v_i^T x_{LS}) v_i \end{aligned}$$

Also, we have

$$\begin{aligned} A^T b &= \sum_{i=1}^{n} v_i \sigma_i u_i^T b \\ &= \sum_{i=1}^{n} v_i \sigma_i (u_i^T b) \\ &= \sum_{i=1}^{n} \sigma_i (u_i^T b) v_i \end{aligned}$$

Comparing the coefficients in the expansions, we have

$$\sigma_i (u_i^T b) = \sigma_i^2 (v_i^T x_{LS}), \qquad i = 1, \ldots, n$$

Hence, we get

$$\begin{aligned} x_{LS} &= \sum_{i=1}^{n} (v_i^T x_{LS}) v_i \\ &= \sum_{i=1}^{n} \frac{u_i^T b}{\sigma_i} v_i. \end{aligned}$$

□

3.4 SVD Analysis of the Degenerate Kernel Problem This exercise illustrates the use of the SVD analysis technique, applied to the problem with a degenerate kernel from Exercise 2.2.

The midpoint quadrature rule plus collocation in the quadrature abscissas lead to a discretized problem with a matrix A whose elements are given by

$$a_{ij} = h\left((i + 2j - \frac{3}{2})h - 3\right), \qquad i, j = 1, 2, \ldots, n$$

with $h = 2/n$. Show that the quadrature abscissas are $t_j = -1 + (j - \frac{1}{2})h$ for $j = 1, \ldots, n$, and verify the above equation for a_{ij}.

Show that the columns of A are related by

$$a_{i,j+1} + a_{i,j-1} = 2a_{i,j}, \qquad i = 1, \ldots, n, \quad j = 2, \ldots, n-1,$$

and, consequently, that the rank of A is 2 for all $n \geq 2$. Verify this experimentally by computing the SVD of A for different values of n.

Since, for this matrix, $A = u_1 \sigma_1 v_1^T + u_2 \sigma_2 v_2^T$, it follows that Ax is always a linear combination of u_1 and u_2 for any x. Show this. Then justify (e.g., by plotting the singular vectors u_1 and u_2) that linear combinations of these vectors represent samples of a linear function, in accordance with the results from Exercise 2.2.

Solution.
For the midpoint rule in the interval $[0, 1]$, we have

$$t_j = \frac{j - \frac{1}{2}}{n}, \qquad \omega_j = \frac{1}{n}, \qquad j = 1, 2, \ldots, n.$$

Then, by mapping from $[0,1]$ to $[-1,1]$, we have

$$t_j = -1 + \frac{(j-\frac{1}{2})2}{n} = -1 + (j - \frac{1}{2})h, \qquad \omega_j = \frac{2}{n} = h, \qquad j = 1, 2, \ldots, n.$$

The matrix A whose elements are given by

$$\begin{aligned} a_{ij} &= \omega_j K(s_i, t_j) \\ &= h(s_i + 2t_j) \\ &= h\left(-1 + (i - \frac{1}{2})h + 2(-1 + (j - \frac{1}{2})h)\right) \\ &= h\left((i + 2j - \frac{3}{2})h - 3\right) \end{aligned}$$

For the column of A, we have

$$\begin{aligned} a_{i,j+1} + a_{i,j-1} &= h\left((i + 2(j+1) - \frac{3}{2})h - 3\right) + h\left((i + 2(j-1) - \frac{3}{2})h - 3\right) \\ &= h\left((2i + 2j - 2 \cdot \frac{3}{2})h - 6\right) \\ &= 2a_{i,j} \end{aligned}$$

Then, we have

$$a_{i,j+1} - a_{i,j} = a_{i,j} - a_{i,j-1}, \qquad i = 1, \ldots, n, \quad j = 2, \ldots, n-1.$$

Let a_j be the jth column of A and vector $e := a_2 - a_1$.Then we have

$$e = a_j - a_{j-1}, \qquad j = 2, \ldots, n.$$

For $j = 2, \ldots, n$, we have

$$a_j = a_1 + (j-1)e$$

Hence, the span of the column vectors is generated by the first column and e. The rank of A is 2 for all $n \geq 2$. Asumme $A = u_1\sigma_1 v_1^T + u_2\sigma_2 v_2^T$, we have

$$Ax = u_1\sigma_1 v_1^T x + u_2\sigma_2 v_2^T x = \sigma_1(v_1^T x)u_1 + \sigma_2(v_2^T x)u_2,$$

it follows that Ax is always a linear combination of u_1 and u_2 for any x. □

3.6 SVD Analysis of a One-Dimensional Image Reconstruction Problem

The purpose of this exercise is to illustrate how the SVD can be used to analyze the smoothing effects of a first-kind Fredholm integral equation. We use the one-dimensional reconstruction test problem, which is implemented in Regularization Tools as function `shaw`. The kernel in this problem is given by

$$K(s,t) = (\cos(s) + \cos(t))^2 \left(\frac{\sin(\pi(\sin(s) + \sin(t)))}{\pi(\sin(s) + \sin(t))}\right)^2,$$

$-\pi/2 \leq s, t \leq \pi/2$, while the solution is

$$f(t) = 2\exp(-6(t-0.8)^2) + \exp(-2(t+0.5)^2).$$

This integral equation models a situation where light passes through an infinitely long slit, and the function $f(t)$ is the incoming light intensity as a function of the incidence angle t. The problem is discretized by means of the midpoint quadrature rule to produce A and x^{exact}, after which the exact right-hand side is computed as $b^{\text{exact}} = Ax^{\text{exact}}$. The elements of b^{exact} represent the outgoing light intensity on the other side of the slit.

Choose $n = 24$ and generate the problem. Then compute the SVD of A, and plot and inspect the left and right singular vectors. What can be said about the number of sign changes in these vectors?

Use the function `picard` from *Regularization Tools* to inspect the singular values σ_i and the SVD coefficients $u_i^T b^{\text{exact}}$ of the exact solution b^{exact}, as well as the corresponding solution coefficients $u_i^T b^{\text{exact}}/\sigma_i$. Is the Picard condition satisfied?

Add a very small amount of noise e to the right-hand side b^{exact}, i.e., $b = b^{\text{exact}} + e$, with $\|e\|_2/\|b^{\text{exact}}\|_2 = 10^{-10}$. Inspect the singular values and SVD coefficients again. What happens to the SVD coefficients $u_i^T b$ corresponding to the small singular values?

Prelude to the next chapter: Recall that the undesired "naive" solution $x = A^{-1}b$ can be written in terms of the SVD as (3.11). Compute the partial sums

$$x_k = \sum_{i=1}^{k} \frac{u_i^T b}{\sigma_i} v_i$$

for $k = 1, 2, \ldots,$ and inspect the vectors x_k . Try to explain the behavior of these vectors.

Solution.

```
[A,b,x] = shaw(24);
[U,S,V] = svd(A);

s=diag(S);
eta = picard(U,s,b);

e=randn(size(b));
e=e/norm(r1)*norm(b)*10^(-10);
be=b+e;
eta = picard(U,s,be);

clear pn
xe=(U(:,1)'*be/s(1))*V(:,1);
pn(1)=norm(x-xe)
for i=2:16
        xe=xe+(U(:,i)'*be/s(i))*V(:,i)
        pn(i)=norm(x-xe)
end
plot(pn)
```

Figure 1,2 show the first nine left singular vectors u_i, v_i for the gravity surveying problem. We see that the singular vectors have more oscillations as the index i increases, i.e., as the corresponding σ_i decrease.

Figure 3 shows that the Picard condition is satisfied. Figure 4 show that the SVD coefficients $u_i^T b$ remain above the noise level.

Figure 5 shows that the norm of error decrease first and attain a minimum. Then it will increase since it is an ill-posed problem and there are small singular values σ_i.

□

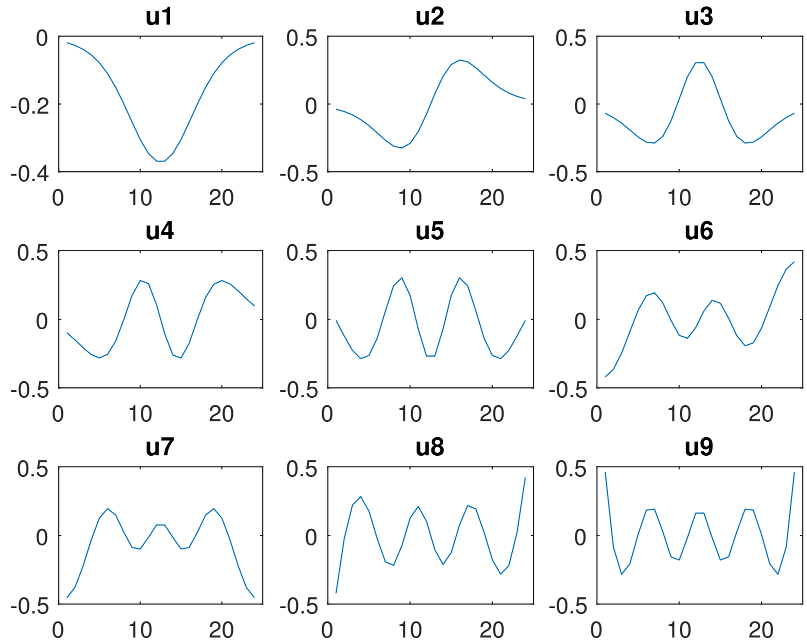

Figure 1: The first 9 left singular vectors u_i

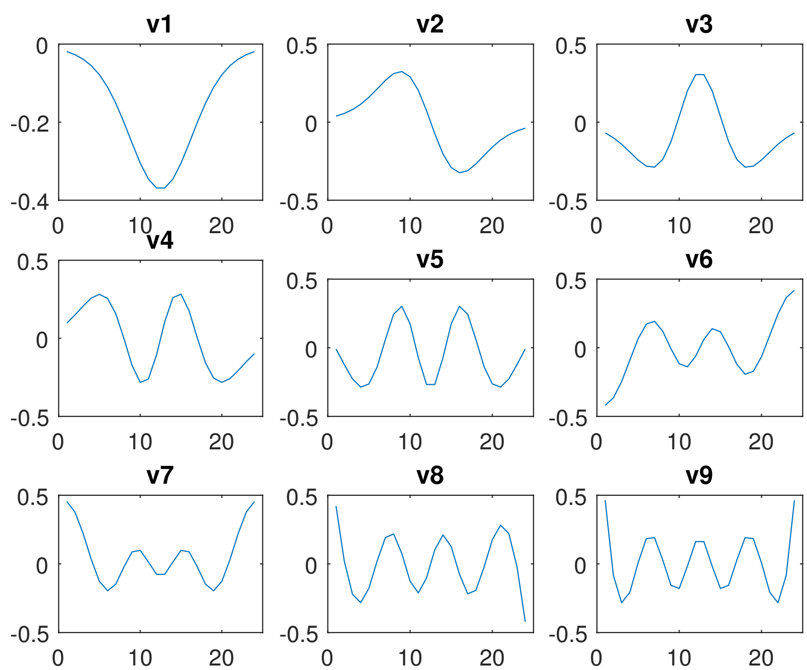

Figure 2: The first 9 left singular vectors v_i

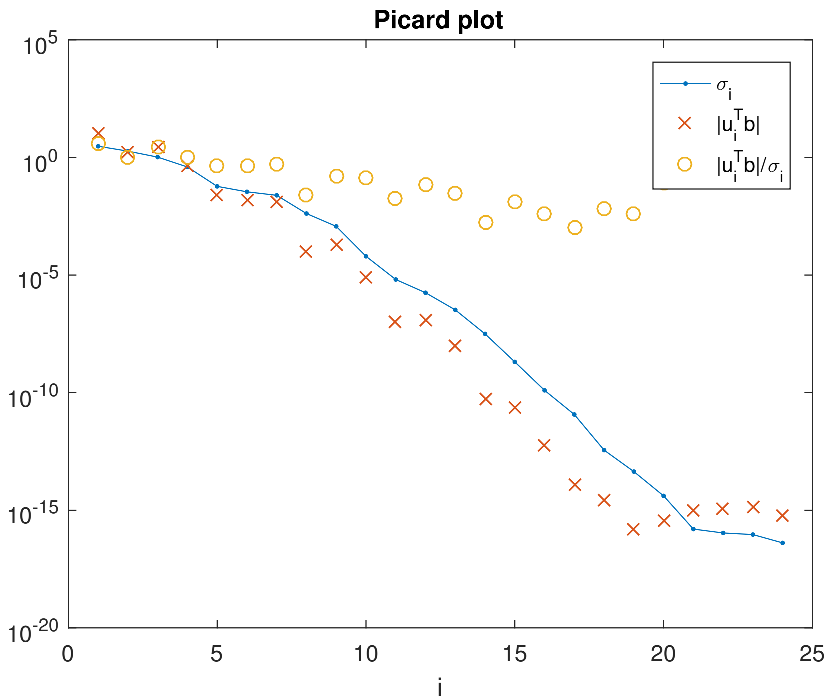

Figure 3: The Picard plots for exact solution

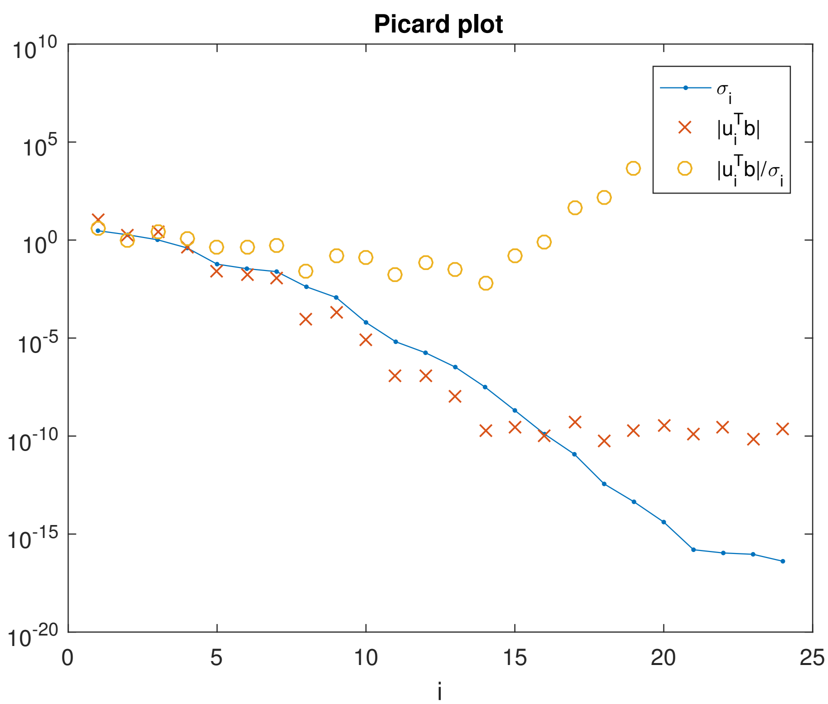

Figure 4: The Picard plots for noisy solution

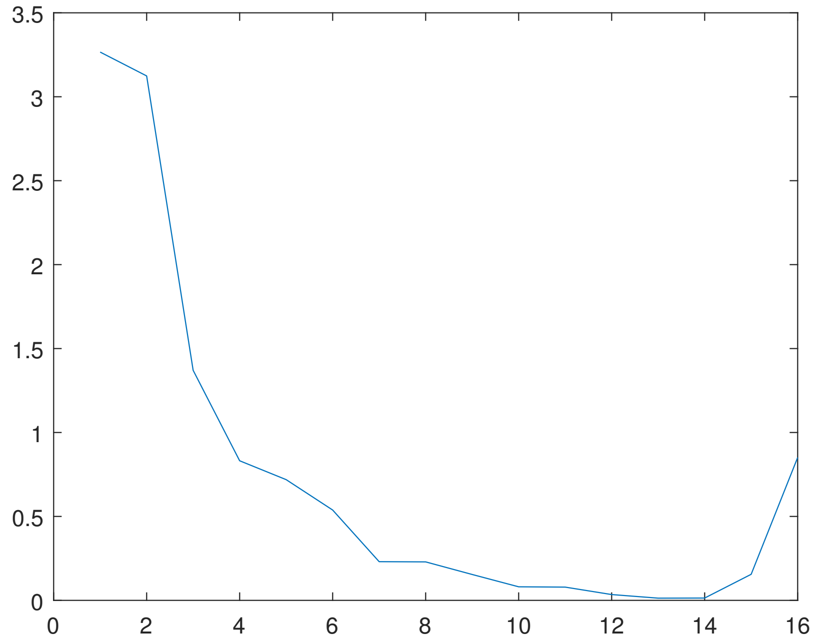

Figure 5: The norm of $x - x_k$