Solution 3

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3. The right-shift operator is defined by $R(a_n) := (0, a_0, a_1, \dots)$. It is an operator that satisfies $\|R\mathbf{x}\| = \|\mathbf{x}\|$ both as $\ell^\infty \to \ell^\infty$ and $\ell^1 \to \ell^1$; it is 1-1 and its image is closed. Note that $LR = I \neq RL$. Show that it is also continuous as $R : \ell^1 \to \ell^\infty$.

   Solution. We only show that $R : \ell^1 \to \ell^\infty$ is continuous. The other properties are easy to check. Since $\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_1$ for any $\mathbf{x} \in \ell^1$, it follows that

$$\|R\mathbf{x}\|_\infty \leq \|R\mathbf{x}\|_1 = \|\mathbf{x}\|_1.$$

   Hence $R : \ell^1 \to \ell^\infty$ is continuous. ◀

4. The mapping $T : \ell^1 \to \ell^1$, defined by $T(a_n) := (a_0, a_1/2, a_2/3, \dots)$, is linear and continuous. It is 1-1, and its image, denoted by $\ell^1_1 := \mathrm{im}T \subset \ell^1$, is not closed in ℓ^1.

   Solution. Clearly T is linear and 1-1. Moreover, T is continuous since

$$\|T\mathbf{x}\| = \sum_{n=0}^{\infty} \left| \frac{x_n}{n+1} \right| \leq \sum_{n=0}^{\infty} |x_n| = \|\mathbf{x}\| \qquad \text{for all } \mathbf{x} = (x_n)_{n=0}^\infty \in \ell^1.$$

   Next we show that its image, $\ell^1_1 := \mathrm{im}T$ is not closed in ℓ^1.
   Let $\mathbf{x}_k := (1, 1/2, \dots, 1/n, 0, 0, 0, \dots) \in \ell^1$. Then

$$T\mathbf{x}_k = (1, 1/2^2, \dots, 1/k^2, 0, 0, 0, \dots) \to \mathbf{y} := (1, 1/2^2, \dots, 1/k^2, 1/(k+1)^2, \dots) \text{ in } \ell^1,$$

   since

$$\|T\mathbf{x}_k - \mathbf{y}\|_1 = \sum_{n=k+1}^{\infty} \frac{1}{n^2} \to 0 \qquad \text{as } k \to \infty.$$

   However, $\mathbf{y} \notin \mathrm{im}T$, hence $\mathrm{im}T$ is not closed. To see this, suppose $\mathbf{y} = T\mathbf{x}$ for some $\mathbf{x} \in \ell^1$, then $\mathbf{x} = (1, 1/2, 1/3, \dots)$ and contradiction arises since

$$\|\mathbf{x}\|_1 = \sum_{n=1}^{\infty} \frac{1}{n} = +\infty.$$

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15. Find, where possible, the norms of the above mentioned operators. For example, $\|\delta_{x_0}\| = 1$ on $C(X)$, and the Volterra operator on $L^\infty[0,1]$ has norm 1.

    Solution. Recall that δ_{x_0} is a linear functional on $C(X)$ defined by $\delta_{x_0}(f) := f(x_0)$. Clearly

$$|\delta_{x_0}(f)| = |f(x_0)| \leq \sup_{x \in X} |f(x)| = \|f\|_{C(X)},$$

which shows that $\|\delta_{x_0}\| \leq 1$. On the other hand, let g be the constant function 1 in $C(X)$. Then

$$1 = |\delta_{x_0}(g)| \leq \|\delta_{x_0}\|\|g\|_{C(X)} = \|\delta_{x_0}\|.$$

Therefore $\|\delta_{x_0}\| = 1$.

Recall that the Volterra operator V is defined by

$$Vf(y) := \int_0^y f.$$

The Volterra operator V is well-defined on $L^\infty[0,1]$ since Vf is measurable and

$$\sup_{y\in[0,1] \text{ a.e.}} |Vf(y)| \leq \sup_{y\in[0,1] \text{ a.e.}} \int_0^y |f(x)|dx \leq \sup_{y\in[0,1] \text{ a.e.}} \int_0^y \|f\|_{L^\infty} dx = \|f\|_{L^\infty} < \infty,$$

so that $Vf \in L^\infty[0,1]$ also. The above inequality also shows that

$$\|Vf\|_{L^\infty} \leq \|f\|_{L^\infty},$$

hence $\|V\|_{L^\infty[0,1]} \leq 1$. On the other hand, if we take $f = 1$ on $[0,1]$, then $\|f\|_{L^\infty} = 1$ and

$$Vf(y) = \int_0^y dx = y,$$

so that $\|Vf\|_{L^\infty} = \sup_{y\in[0,1] \text{ a.e.}} |y| = 1$. Now

$$1 = \|Vf\|_{L^\infty} \leq \|V\|_{L^\infty[0,1]}\|f\|_{L^\infty} \leq \|V\|_{L^\infty[0,1]} = 1.$$

Therefore $\|V\|_{L^\infty[0,1]} = 1$.

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