Solution to Assignment 8

1. The Fourier transform maps $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}(\mathbb{R})$.

   Solution. Note that if $f \in \mathcal{S}(\mathbb{R})$, then its Fourier transform $\hat{f}$ is bounded. Furthermore, for $j, k \geq 0$, the expression
   $$\xi^k \left(\frac{d}{d\xi}\right)^j \hat{f}(\xi)$$
   is bounded, since it is the Fourier transform of
   $$\frac{1}{i^k}\left(\frac{d}{dx}\right)^k [(-ix)^j f(x)],$$
   by Proposition 8.2 (d) and (e). Therefore $\hat{f} \in \mathcal{S}(\mathbb{R})$.

2. Show that the Fourier transform of $e^{-a|x|}$ is equal to $2a/(\xi^2 + a^2)$.

   Solution.
   $$\begin{aligned}
   \mathcal{F}(e^{-a|x|})(\xi) &= \int_{-\infty}^{\infty} e^{-a|x|} e^{-i\xi x} dx \\
   &= \int_0^{\infty} e^{-ax} e^{-i\xi x} dx + \int_{-\infty}^{0} e^{ax} e^{-i\xi x} dx \\
   &= \frac{1}{-a - i\xi} e^{-a-i\xi x}\Big|_0^{\infty} + \frac{1}{a - i\xi} e^{a-i\xi x}\Big|_{-\infty}^{0} \\
   &= \frac{1}{a + i\xi} + \frac{1}{a - i\xi} \\
   &= \frac{2a}{a^2 + \xi^2}.
   \end{aligned}$$

3. Show that the Fourier transform of $\dfrac{\sin x}{x}$ is the function $F(x) = \pi, x \in (-1, 1)$ and $F(x) = 0$ outside $(-1, 1)$. Hint: Use the inversion theorem formally.

   Solution.
   $$\begin{aligned}
   \mathcal{G}(F)(\xi) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\xi) e^{i\xi x} d\xi \\
   &= \frac{1}{2\pi} \int_{-1}^{1} \pi e^{i\xi x} d\xi \\
   &= \frac{1}{2ix}(e^{ix} - e^{-ix}) \\
   &= \frac{\sin x}{x}.
   \end{aligned}$$
   By using the Fourier inversion formally, we conclude that $\mathcal{F}(\frac{\sin x}{x}) = F$.

4. Let $R^2(\mathbb{R})$ be the vector space of all functions f on the real line whose square is improperly integrable. Show that $R^2(\mathbb{R})$ is not a subset of $R(\mathbb{R})$ and $R(\mathbb{R})$ is not a subset of $R^2(\mathbb{R})$. This is in contrast with Riemann integration in which the square of any integrable function is integrable and the square root of any integrable function is integrable. Hint: Consider the function $f(x) = 1/\sqrt{x}$ and $g(x) = 1/x, x \geq 1$.

Solution. Let

$$f(x) = \begin{cases} \dfrac{1}{\sqrt{x}}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad g(x) = \begin{cases} \dfrac{1}{x}, & x \geq 1 \\ 0, & \text{otherwise.} \end{cases}$$

Now

$$\int_{-\infty}^{\infty} |f(x)| dx = \int_0^1 \frac{1}{\sqrt{x}} dx = 2,$$

while

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_0^1 \frac{1}{x} dx = +\infty.$$

Hence $f \in R(\mathbb{R})$ but $f \notin R^2(\mathbb{R})$.

On the other hand,

$$\int_{-\infty}^{\infty} |g(x)| dx = \int_1^{\infty} \frac{1}{x} dx = +\infty,$$

while

$$\int_{-\infty}^{\infty} |g(x)|^2 dx = \int_1^{\infty} \frac{1}{x^2} dx = 1.$$

Hence $g \in R^2(\mathbb{R})$ but $g \notin R(\mathbb{R})$.