Solution 2

Hand in Ex 2.2, no. 4, Ex 2.3 no. 2c by Jan 31.

Ex 2.2

2. 6: The series converges to the value $\frac{1+(-1)}{2} = 0$ at the points of discontinuity $\theta = 0$ and $\theta = \pi$.

   7: The series converges to the value $\frac{1+0}{2} = \frac{1}{2}$ at the points of discontinuity $\theta = 0$ and $\theta = \pi$.

   12: The series converges to the value $\frac{(2a)^{-1}+0}{2} = \frac{1}{4a}$ at the points of discontinuity $\theta = -a$ and $\theta = a$.

   18: The series converges to the value $\frac{e^{b\pi}+e^{-b\pi}}{2} = \cosh b\pi$ at the point of discontinuity $\theta = \pi$.

4. By Theorem 2.1 and entry 16 in Table 1, we have

$$\theta^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\theta \qquad \text{for } -\pi < \theta < \pi.$$

Putting $\theta = \pi$, we have $\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}(-1)^n$, and hence

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Putting $\theta = 0$, we have $0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$, and hence

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

6. By Theorem 2.1 and entry 18 in Table 1, we have

$$e^{b\theta} = \frac{\sinh b\pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{b - in} e^{in\theta} \qquad \text{for } -\pi < \theta < \pi.$$

Putting $\theta = 0$, we have $1 = \frac{\sinh b\pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{b-in}$, and hence

$$\begin{aligned} \frac{\pi}{\sinh b\pi} &= \frac{1}{b} + \sum_{n=1}^{\infty} \left(\frac{(-1)^n}{b - in} + \frac{(-1)^{-n}}{b + in} \right), \\ &= \frac{1}{b} + 2b \sum_{n=1}^{\infty} \frac{(-1)^n}{b^2 + n^2}. \end{aligned}$$

Thus

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{b^2 + n^2} = \frac{\pi}{2b} \text{csch } b\pi - \frac{1}{2b^2}.$$

Ex 2.3

1. From (2.17), we have
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n\theta = \frac{\theta}{2} \qquad \text{for } -\pi < \theta < \pi.$$
By Theorem 2.4, we have, for $-\pi < \theta < \pi$,
$$\frac{\theta^2}{4} = \int_0^\theta \frac{\phi}{2}\, d\phi = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\theta^2}{4}\, d\theta + \sum_{n=1}^{\infty} -\frac{(-1)^{n+1}/n}{n} \cos n\theta$$
so that
$$\theta^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\theta.$$

2. From entry 16 of Table 1 and Theorem 2.1,
$$\theta^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\theta.$$
   (a) Using Theorem 2.4, we have
$$\frac{\theta^3}{3} - \frac{\pi^2\theta}{3} = \int_0^\theta (\phi^2 - \frac{\pi^2}{3})\, d\phi = C_0 + 4\sum_{n=1}^{\infty} \frac{(-1)^n/n^2}{n} \sin n\theta,$$
   where
$$C_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} (\frac{\theta^3}{3} - \frac{\pi^2\theta}{3})\, d\theta = 0.$$
   Hence
$$\theta^3 - \pi^2\theta = 12\sum_{n=1}^{\infty} \frac{(-1)^n \sin n\theta}{n^3},$$
   whenever $-\pi \le \theta \le \pi$.

   (b) Applying Theorem 2.4 to the formula in (a), we have
$$\frac{\theta^4}{4} - \frac{\pi^2\theta^2}{2} = \int_0^\theta (\phi^3 - \pi^2\phi)\, d\phi = C_0' + 12\sum_{n=1}^{\infty} -\frac{(-1)^n/n^3}{n} \cos n\theta,$$
   where
$$C_0' = \frac{1}{2\pi}\int_{-\pi}^{\pi} (\frac{\theta^4}{4} - \frac{\pi^2\theta^2}{2})\, d\theta = -\frac{7\pi^4}{60}.$$
   Hence
$$\theta^4 - 2\pi^2\theta^2 = 48\sum_{n=1}^{\infty} \frac{(-1)^{+1}n}{n^4} \cos n\theta - \frac{7\pi^4}{15},$$
   whenever $-\pi \le \theta \le \pi$.

   (c) By putting $\theta = \pi$ in the formula in (b), the formula $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$ follows.

Supplementary Exercise. Establish the following summation formula:

$$\sum_{k=0}^{n} \sin(k+1/2)x = \frac{\sin^2 \frac{n+1}{2} x}{\sin x/2}.$$

By compound angle formulas,

$$\cos kx = \cos(k+\frac{1}{2}-\frac{1}{2})x = \cos(k+\frac{1}{2})\cos\frac{x}{2} + \sin(k+\frac{1}{2})\sin\frac{x}{2}.$$

and

$$\cos(k+1)x = \cos(k+\frac{1}{2}+\frac{1}{2})x = \cos(k+\frac{1}{2})\cos\frac{x}{2} - \sin(k+\frac{1}{2})\sin\frac{x}{2}.$$

Hence

$$2\sin(k+\frac{1}{2})x\sin\frac{x}{2} = \cos kx - \cos(k+1)x.$$

Summing both sides over k, we have

$$2\left(\sum_{k=0}^{n} \sin(k+\frac{1}{2})x\right)\sin\frac{x}{2} = 1-\cos(n+1)x = 2\sin^2\left(\frac{n+1}{2}\right).$$

The result follows.