Homework # 4

Due: Mar. 24, 2011, 4:30 PM

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1. Three dice are rolled. By assuming that each of the $6^3 = 216$ possible outcomes is equally likely, find the probabilities attached to the possible values that X can take on, where X is the sum of the 3 dice.

Solution:
$P\{X = 3\} = \frac{1}{216}$, outcome $\{1, 1, 1\}$ $P\{X = 4\} = \frac{3}{216}$, outcome {2,1,1},{1,2,1},{1,1,2}

$P\{X = 4\} = \frac{3}{216}$, outcome {2,1,1},{1,1,2},{1,2,1}

$P\{X = 5\} = \frac{6}{216}$, outcome {2,2,1},{2,1,2},{1,2,2}, {3,1,1},{1,3,1},{1,1,3}

$P\{X = 6\} = \frac{10}{216}$, outcome {2,2,2}, {4,1,1},{1,4,1},{1,1,4}, {1,2,3},{1,3,2},{2,1,3},{2,3,1},{3,2,1},{3,1,2}

$P\{X = 7\} = \frac{15}{216}$, outcome {5,1,1},{1,5,1},{1,1,5}, {4,1,2},{4,2,1},{2,1,4},{2,4,1},{1,2,4},{1,4,2}
{3,3,1},{3,1,3},{1,3,3}, {3,2,2},{2,3,2},{2,2,3}

$P\{X = 8\} = \frac{21}{216}$, outcome {6,1,1},{1,6,1},{1,1,6}, {5,1,2},{5,2,1},{2,1,5},{2,5,1},{1,2,5},{1,5,2}
{3,3,2},{3,2,3},{2,3,3}, {4,2,2},{2,4,2},{2,2,4}, {4,1,3},{4,3,1},{3,1,4},{3,4,1},{1,3,4},{1,4,3}

$P\{X = 9\} = \frac{25}{216}$, outcome $\{6, 1, 2\}$, $\{6, 2, 1\}$, $\{2, 1, 6\}$, $\{2, 6, 1\}$, $\{1, 2, 6\}$, $\{1, 6, 2\}$, $\{5, 2, 2\}$, $\{2, 5, 2\}$, $\{2, 2, 5\}$
$\{5, 1, 3\}$, $\{5, 3, 1\}$, $\{3, 1, 5\}$, $\{3, 5, 1\}$, $\{1, 3, 5\}$, $\{1, 5, 3\}$, $\{1, 4, 4\}$, $\{4, 4, 1\}$, $\{4, 1, 4\}$
{4,2,3},{4,3,2},{3,2,4},{3,4,2},{2,3,4},{2,4,3}, {3,3,3}

$P\{X = 10\} = \frac{27}{216}$, outcome {6,2,2},{2,6,2},{2,2,6}, {6,1,3},{6,3,1},{3,1,6},{3,6,1},{1,3,6},{1,6,3}
{5,2,3},{5,3,2},{3,2,5},{3,5,2},{2,3,5},{2,5,3}, {5,1,4},{5,4,1},{4,1,5},{4,5,1},{1,4,5},{1,5,4}
{2,4,4},{4,4,2},{4,2,4}, {3,3,4},{4,3,3},{3,4,3}

$P\{X = 11\} = \frac{27}{216}$, outcome {6,2,3},{6,3,2},{3,2,6},{3,6,2},{2,3,6},{2,6,3}, {6,1,4},{6,4,1},{4,1,6}
{4,6,1},{1,4,6},{1,6,4}, {5,2,4},{5,4,2},{4,2,5},{4,5,2},{2,4,5},{2,5,4}, {5,3,3},{3,3,5},{3,5,3}
{5,5,1},{5,1,5},{1,5,5}, {3,4,4},{4,3,4},{4,4,3}

$P\{X = 12\} = \frac{25}{216}$, outcome {6,2,4},{6,4,2},{4,2,6},{4,6,2},{2,4,6},{2,6,4}, {6,1,5},{6,5,1},{5,1,6}
{5,6,1},{1,5,6},{1,6,5}, {6,3,3},{3,3,6},{3,6,3} {5,5,2},{5,2,5},{2,5,5}
{5,3,4},{5,4,3},{4,3,5},{4,5,3},{3,4,5},{3,5,4}, {4,4,4}

$P\{X = 13\} = \frac{21}{216}$, outcome {6,6,1},{1,6,6},{6,1,6}, {6,2,5},{6,5,2},{5,2,6},{5,6,2},{2,5,6},{2,6,5}
{6,3,4},{6,4,3},{4,3,6},{4,6,3},{3,4,6},{3,6,4}, {5,5,3},{3,5,5},{5,3,5}, {5,4,4},{4,5,4},{4,4,5}

$P\{X = 14\} = \frac{15}{216}$, outcome {6,6,2},{2,6,6},{6,2,6}, {6,3,5},{6,5,3},{5,3,6},{5,6,3},{3,5,6},{3,6,5}
{6,4,4},{4,6,4},{4,4,6}, {5,5,4},{4,5,5},{5,4,5}

$P\{X = 15\} = \frac{10}{216}$, outcome {6,6,3},{3,6,6},{6,3,6}, {6,4,5},{6,5,4},{5,4,6},{5,6,4},{4,5,6},{4,6,5}
{5,5,5}

$P\{X = 16\} = \frac{6}{216}$, outcome {6,6,4},{4,6,6},{6,4,6}, {6,5,5},{5,5,6},{5,6,5}

$P\{X = 17\} = \frac{3}{216}$, outcome {6,6,5},{5,6,6},{6,5,6}

$P\{X = 18\} = \frac{1}{216}$, outcome {6,6,6}

2. Let X represent the difference between the number of heads and the number of tails obtained when a coin is tossed n times. What are the possible values of X?

   Solution:
   Let i denotes the number of heads and j denotes the number of tails when a coin is tossed n times.
   Since $i + j = n$, so $j = n - i$ or $i = n - j$.
   Thus $X = i - j = 2i - n, i = 0, \ldots, n$
   Or $X = n - 2j, j = 0, \ldots, n$
   Note: both answers are right, you only need write one of them

3. Five distinct numbers are randomly distributed to players numbered 1 through 5. Whenever two players compare their numbers, the one with the higher one is declared the winner. Initially, players 1 and 2 compare their numbers; the winner then compares her number with that of player 3, and so on. Let X denote the number of times player 1 is a winner. Find $P\{X = i\}, i = 0, 1, 2, 3, 4$

   Solution:
   Without loss of generality, we can sort these five numbers and use $x_1, \ldots, x_5$ to denote these sorted numbers, such that $x_1 < x_2 < \ldots < x_5$. Let X_i denote the number distributed to player i, $i = 1, \ldots, 5$.

   (a) Compute $P\{X = 0\}$

$$\begin{aligned} P\{X = 0\} &= P\{\text{player 1 loses to player 2}\} \\ &= P\{X_1 < X_2\} \\ &= \sum_{i=1}^{5} P\{X_1 < X_2 | X_1 = x_i\} P\{X_1 = x_i\} \\ &= \sum_{i=1}^{5} \frac{5-i}{4}\frac{1}{5} \\ &= \frac{1}{2} \end{aligned}$$

   (b) Compute $P\{X = 1\}$

$$\begin{aligned} P\{X = 1\} &= P\{\text{player 1 wins player 2 and player 1 loses to player 3}\} \\ &= P\{X_2 < X_1 < X_3\} \\ &= \sum_{i=1}^{5} P\{X_2 < X_1 < X_3 | X_1 = x_i\} P\{X_1 = x_i\} \\ &= P\{X_2 < X_1 < X_3 | X_1 = x_1\} P\{X_1 = x_1\} + \sum_{i=2}^{5} P\{X_2 < X_1 < X_3 | X_1 = x_i\} P\{X_1 = x_i\} \\ &= 0 + \sum_{i=2}^{5} \frac{i-1}{4}\frac{5-i}{3}\frac{1}{5} \\ &= \frac{3+4+3}{60} \\ &= \frac{1}{6} \end{aligned}$$

(c) Compute $P\{X = 2\}$

$$\begin{aligned} P\{X=2\} &= P\{\text{player 1 wins player 2 and players 3 and player 1 loses to player 4}\} \\ &= P\{X_2 < X_1, X_3 < X_1, X_1 < X_4\} \\ &= \sum_{i=1}^{5} P\{X_2 < X_1, X_3 < X_1, X_1 < X_4 | X_1 = x_i\} P\{X_1 = x_i\} \end{aligned}$$

Since $P\{X_2 < X_1, X_3 < X_1, X_1 < X_4 | X_1 = x_i\} = 0$, when $i = 1, 2$. So,

$$\begin{aligned} P\{X=2\} &= \sum_{i=3}^{5} P\{X_2 < X_1, X_3 < X_1, X_1 < X_4 | X_1 = x_i\} P\{X_1 = x_i\} \\ &= \sum_{i=3}^{5} \frac{i-1}{4}\frac{i-2}{3}\frac{5-i}{2}\frac{1}{5} \\ &= \frac{4+6}{120} \\ &= \frac{1}{12} \end{aligned}$$

(d) Compute $P\{X = 3\}$

$$\begin{aligned} P\{X=3\} &= P\{\text{player 1 wins player 2, players 3 and player 4, and player 1 loses to player 5}\} \\ &= P\{\text{player 5 has the largest number and player 1 has the second largest number}\} \\ &= \frac{1}{5} \times \frac{1}{4} \\ &= \frac{1}{20} \end{aligned}$$

(e) Compute $P\{X = 4\}$

$$\begin{aligned} P\{X=4\} &= P\{\text{player 1 wins all the other players}\} \\ &= P\{\text{player 1 has the largest number}\} \\ &= \frac{1}{5} \end{aligned}$$

4. Suppose that the distribution function of X is given by

$$y = \begin{cases} 0 & b < 0 \\ b/4 & 0 \le b < 1 \\ 1/2 + (b-1)/4 & 1 \le b < 2 \\ 11/12 & 2 \le b < 3 \\ 1 & 3 \le b \end{cases}$$

(a) Find $P\{X = i\}, i = 1, 2, 3$

(b) Find $P\{\frac{1}{2} < X < \frac{3}{2}\}$

Solution:

(a) Find $P\{X = i\}, i = 1, 2, 3$

$P\{X = 1\} = \frac{1}{2} + \frac{1-1}{4} - \frac{1}{4} = \frac{1}{4}$

$P\{X = 2\} = \frac{11}{12} - \frac{1}{2} - \frac{2-1}{4} = \frac{1}{6}$

$P\{X = 3\} = 1 - \frac{11}{12} = \frac{1}{12}$

(b) Find $P\{\frac{1}{2} < X < \frac{3}{2}\}$

$P\{\frac{1}{2} < X < \frac{3}{2}\} = P\{X < \frac{3}{2}\} - P\{X \leq \frac{1}{2}\} = \frac{1}{2} + \frac{3/2-1}{4} - \frac{1/2}{4} = \frac{1}{2}$

5. Four independent flips of a fair coin are made. Let X denote the number of heads obtained. Plot the probability mass function of the random variable $X - 2$.

Solution:
Let $Y = X - 2$. Since $0 \leq X \leq 4$, so $-2 \leq Y \leq 2$

$P\{Y = -2\} = P\{X = 0\} = \binom{4}{0}(\frac{1}{2})^0(\frac{1}{2})^{4-0} = \frac{1}{16}$

$P\{Y = -1\} = P\{X = 1\} = \binom{4}{1}(\frac{1}{2})^1(\frac{1}{2})^{4-1} = \frac{1}{4}$

$P\{Y = 0\} = P\{X = 2\} = \binom{4}{2}(\frac{1}{2})^2(\frac{1}{2})^{4-2} = \frac{3}{8}$

$P\{Y = 1\} = P\{X = 3\} = \binom{4}{3}(\frac{1}{2})^3(\frac{1}{2})^{4-3} = \frac{1}{4}$

$P\{Y = 2\} = P\{X = 4\} = \binom{4}{4}(\frac{1}{2})^4(\frac{1}{2})^{4-4} = \frac{1}{16}$

Then it is straight to plot the probability mass function of the random variable $X - 2$. I just omit the figure here.

6. Four buses carrying 148 students from the same school arrive at a football stadium. The buses carry, respectively, 40, 33, 25 and 50 students. One of the students is randomly selected. Let X denote the number of students that were on the bus carrying the randomly selected student. One of the 4 bus drivers is also randomly selected. Let Y denote the number of students on her bus.

(a) Which of $E[X]$ or $E[Y]$ do you think is larger? Why?
(b) Compute $E[X]$ and $E[Y]$.

Solution:

(a) Which of $E[X]$ or $E[Y]$ do you think is larger? Why?

$E[X]$ is larger, since the driver selected is equally likely to be from any of the 4 buses, and the student selected is more likely to have come from a bus carrying a large number of students.

(b) Compute $E[X]$ and $E[Y]$.

$P\{X = i\} = \frac{i}{148}$, where $i = 40, 33, 25, 50$

$E[X] = 40 \times \frac{40}{148} + 33 \times \frac{33}{148} + 25 \times \frac{25}{148} + 50 \times \frac{50}{148} \approx 39.28$

$E[Y] = \frac{40+33+25+50}{4} = 37$

7. Two coins are to be flipped. The first coin will land on heads with probability 0.6, the second with probability 0.7. Assume that the results of the flips are independent, and let X equal the total number of heads that result.

   (a) Find $P\{X = 1\}$.

   (b) Determine $E[X]$.

   Solution:

   (a) Find $P\{X = 1\}$.

   $$P\{X = 1\} = 0.6 \times 0.3 + 0.4 \times 0.7 = 0.46$$

   (b) Determine $E[X]$.

   $$P\{X = 2\} = 0.6 \times 0.7 = 0.42$$

   $$E[X] = 1 \times 0.46 + 2 \times 0.42 = 1.3$$