Homework # 3 Solutions

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1. We define (m, n) as the result of one roll, i.e., the first dice lands on m and the second dice lands on n. Obviously, we have $P\{(m,n)\} = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$ for $m = 1, ..., 6$ and $n = 1, ..., 6$.

$$\begin{aligned} P\{\text{the first dice lands on 6}|\text{the sum is } i\} &= P\{m = 6|m + n = i\} \\ &= \frac{P\{m = 6, n = i - 6\}}{P\{m + n = i\}} \end{aligned} \quad (1)$$

Based on Equation (1), we have:

- $i = 2, 3, 4, 5, 6$: $\frac{P\{m=6,n=i-6\}}{P\{m+n=i\}} = 0$;
- $i = 7$: $\frac{P\{m=6,n=i-6\}}{P\{m+n=i\}} = \frac{P\{(6,1)\}}{P\{(6,1)\}+P\{(1,6)\}+P\{(2,5)\}+P\{(5,2)\}+P\{(3,4)\}+P\{(4,3)\}} = \frac{\frac{1}{36}}{\frac{1}{36}\times 6} = \frac{1}{6}$;
- $i = 8$: $\frac{P\{m=6,n=i-6\}}{P\{m+n=i\}} = \frac{P\{(6,2)\}}{P\{(6,2)\}+P\{(2,6)\}+P\{(3,5)\}+P\{(5,3)\}+P\{(4,4)\}} = \frac{\frac{1}{36}}{\frac{1}{36}\times 5} = \frac{1}{5}$;
- $i = 9$: $\frac{P\{m=6,n=i-6\}}{P\{m+n=i\}} = \frac{P\{(6,3)\}}{P\{(6,3)\}+P\{(3,6)\}+P\{(4,5)\}+P\{(5,4)\}} = \frac{\frac{1}{36}}{\frac{1}{36}\times 4} = \frac{1}{4}$;
- $i = 10$: $\frac{P\{m=6,n=i-6\}}{P\{m+n=i\}} = \frac{P\{(6,4)\}}{P\{(6,4)\}+P\{(4,6)\}+P\{(5,5)\}} = \frac{\frac{1}{36}}{\frac{1}{36}\times 3} = \frac{1}{3}$;
- $i = 11$: $\frac{P\{m=6,n=i-6\}}{P\{m+n=i\}} = \frac{P\{(6,5)\}}{P\{(6,5)\}+P\{(5,6)\}} = \frac{\frac{1}{36}}{\frac{1}{36}\times 2} = \frac{1}{2}$;
- $i = 12$: $\frac{P\{m=6,n=i-6\}}{P\{m+n=i\}} = \frac{P\{(6,6)\}}{P\{(6,6)\}} = \frac{\frac{1}{36}}{\frac{1}{36}\times 1} = 1$;

2.

$$\begin{aligned} P\{\text{both are girls}|\text{the older one is a girl}\} &= \frac{P\{\text{both are girls and the older one is a girl}\}}{P\{\text{ the older one is a girl}\}} \\ &= \frac{P\{\text{the older one is a girl, and the younger one is also a girl}\}}{P\{\text{ the older one is a girl}\}} \\ &= \frac{\frac{1}{2}\times\frac{1}{2}}{\frac{1}{2}} \\ &= \frac{1}{2} \end{aligned} \quad (2)$$

3. We use A, B, and C to denote the ball chosen from urn A, urn B, and urn C respectively. We use w to represent *white* and r to represent *red*. We have:

$$\begin{aligned} & P\{\text{the ball chosen from URN } A \text{ was white}|\text{exactly 2 white balls were selected}\} \\ = \; & P\{A = w|\text{exactly } 2w\} \\ = \; & \frac{P\{A = w, B = w, c = r\} + P\{A = w, B = r, c = w\}\}}{P\{A = w, B = w, c = r\} + P\{A = w, B = r, c = w\} + P\{A = r, B = w, c = w\}} \\ = \; & \frac{\frac{2}{6}\times\frac{8}{12}\times\frac{3}{4}+\frac{2}{6}\times\frac{4}{12}\times\frac{1}{4}}{\frac{2}{6}\times\frac{8}{12}\times\frac{3}{4}+\frac{2}{6}\times\frac{4}{12}\times\frac{1}{4}+\frac{4}{6}\times\frac{8}{12}\times\frac{1}{4}} \\ = \; & \frac{7}{11} \end{aligned} \quad (3)$$

4. (a) $P\{\text{the first 2 balls are black, and the next 2 are white}\} = \frac{7}{12} \times \frac{9}{14} \times \frac{5}{16} \times \frac{7}{18} = \frac{35}{768}$

   (b) We use w and b to denote *white* and *black* respectively, and use a string to represent the colors of the first four balls, e.g., $wwbb$ means that the first ball selected is white, the second ball selected is white, the third ball selected is black, and the fourth ball selected is black.

$$\begin{aligned}
& P\{\text{ of the first 4 balls selected, exactly 2 are black}\} \\
= \; & P\{bbww\} + P\{bwbw\} + P\{bwwb\} + P\{wbbw\} + P\{wbwb\} + P\{wwbb\} \\
= \; & \frac{7}{12}\frac{9}{14}\frac{5}{16}\frac{7}{18} + \frac{7}{12}\frac{5}{14}\frac{9}{16}\frac{7}{18} + \frac{7}{12}\frac{5}{14}\frac{7}{16}\frac{9}{18} + \frac{5}{12}\frac{7}{14}\frac{9}{16}\frac{7}{18} + \frac{5}{12}\frac{7}{14}\frac{7}{16}\frac{9}{18} + \frac{5}{12}\frac{7}{14}\frac{7}{16}\frac{9}{18} \\
= \; & \frac{35}{128}
\end{aligned} \tag{4}$$

Note: "it is replaced in the urn along with 2 other balls of the same color" means that *keep the selected ball in the urn and add two additional balls with the same color in the urn.*

5. We use D to denote the event "a family owns a dog", and use C to denote the even " a family owns a cat". We have $P(D) = 0.36$, $P(C) = 0.30$, and $P(C|D) = 0.22$.

   (a) $P\{\text{A randomly selected family owns both a dog and a cat}\}$
   $= P(CD) = P(D)P(C|D) = 0.36 \times 0.22 = 0.0792$

   (b) $P\{\text{a randomly selected family owns a dog given that it owns a cat}\}$
   $= P(D|C) = \frac{P(CD)}{P(C)} = \frac{0.0792}{0.30} = 0.264$

6. We use F to denote the event "the student is female", and use CS to denote the event "the student is majoring in computer science". We have $P(F) = 0.52$, $P(CS) = 0.05$, and $P(F, CS) = 0.02$.

   (a) $P\{\text{a student is female| the student is majoring in computer science}\}$
   $= P(F|CS) = \frac{P(F,CS)}{P(CS)} = \frac{0.02}{0.05} = 0.40$

   (b) $P\{\text{a student is majoring in computer science| the student is female}\}$
   $= P(CS|F) = \frac{P(F,CS)}{P(F)} = \frac{0.02}{0.52} = \frac{1}{26} \approx 0.0385$