Homework # 2 Solutions

Instructor: John C.S. Lui

1. Solution:
   (a) B is a subset of C. Hence $B \bigcup C = C$.
   (b) Hence $B \bigcap C = B$.
   (c) $A \bigcup C = S$, the universal event.
   (d) $A \bigcap C =$ null event.
2. Solution:
   The sample space consists of 64 6-tuples of 0's and 1's. Enumerating these 6-tuples, we determine that the cardinality of event and corresponding probability as follows:
   (a) $|A| = 56$, $P(A) = \frac{56}{64} = \frac{7}{8}$,
   (b) $|B| = 41$, $P(B) = \frac{41}{64}$,
   (c) $|C| = 32$, $P(C) = \frac{1}{2}$,
   (d) $|A \bigcup B \bigcup C| = 58$, $P(A \bigcup B \bigcup C) = \frac{29}{32}$.
3. Solution:
   (a) If we can use the same digit again, we can construct $5 \times 2 = 10$ even digits.
   (b) If we can not use the same digit, we can construct $3 \times 2 + 2 \times 1 = 8$ even digits.
4. Solution:
   When k is different, we have different probability.
   (a) When $k = 0$, $P(1) = \frac{5^3}{10^3} = 0.125$.
   (b) When $k = 1$, $P(1) = \frac{3\times5\times5^2}{10^3} = 0.375$.
   (c) When $k = 2$, $P(1) = \frac{3\times5\times5^2}{10^3} = 0.375$.
   (d) When $k = 3$, $P(1) = \frac{5\times5^2}{10^3} = 0.125$.

   (BTW: here it is assumed that three-digit numbers include 0-999. You can also assume that they only include 100-999.)
5. Solution:
   $|S| = \binom{15}{3} = 455$, $|E| = \binom{5}{3} = 10$
   $P(E) = \frac{|E|}{|S|} = \frac{2}{91} = 2.1978 \times 10^{-2}$.
6. Solution:
   The sample space is $S = \{(x_1, x_2, x_3, x_4, x_5) : x_i \in \{1, ..., 356\}\}$, $|S| = 356^5$
   Define event $E =$ at least two have same birthday, $\overline{E} =$ none have the same birthday.
   $|\overline{E}| = P(365, 5) = \frac{365!}{360!}$
   $P(E) = 1 - P(\overline{E}) = 1 - \frac{365!}{360! 365^5} = 0.0272$.

7. Solution:

   The sample space is $S = \{(i_1, i_2, ..., i_n) : i_j \in \{1, 2, ...n\}\}$. Hence $|S| = n^n$.

   Let n^n possible arrangement be equally likely. Compute the probability that only one processor is empty. First we compute the probability that processor labelled 1 is idle. Let A_1 be the event of interest. Then n jobs are distributed such that none of the $n-1$ processors are empty and hence all except one processor hold 1 job each.

   Let B_j be the event that processor j has two jobs, $B_j = \{(i_1, i_2, ..., i_n) : i_k \in \{2, ..., n\}, i_{k_1} = i_{k_2} = j\}$

   Then, $A_1 = \bigcap_{j=2}^{n} B_j$, $P(B_j) = \frac{\binom{n}{2}(n-2)!}{n^n}$.

   $P(A_1) = \frac{(n-1)(n-2)!\binom{n}{2}}{n^n} = \frac{n!}{n^n}\frac{n-1}{2}$

   Now the probability of exactly 1 processor idle $= \bigcap_{i=1}^{n} A_i = \frac{n-1}{2}\frac{(n-1)!}{n^{n-2}}$.

8. Solution:

   The shortest path must of be length $m+n$. In each step, we should choose go up or go left. So the number of shortest path is $\binom{m+n}{n}$.