ENGG 2440A: Discrete Mathematics for Engineers — Homework 5
The Chinese University of Hong Kong, Fall 2017 — due Thursday 23 November

Questions 1 to 6 are worth 10 points each. Please turn in solutions to *four* questions of your choice. Write your solutions clearly and concisely. If you do not explain your answer you will be given no credit. You are encouraged to collaborate on the homework, but you must write your own solutions and give credit to your collaborators on your solution sheet. Copying someone's solution or pasting material you found online without reference will be considered plagiarism and may result in failing the whole course.

Questions

1. Find exact closed-form solutions to the following recurrences.
   (a) $T(n) = 3T(n/2) + n$, $T(1) = 1$, where n is a power of 2.
   (b) $F(n) = \frac{1}{3}F(n-1) + n$, $F(0) = 0$.
   (c) $f(n) = 8f(n-1) - 15f(n-2)$, $f(0) = 0$, $f(1) = 1$.
   (d) $f(n) = f(n-1) + f(n-2) + 1$, $f(0) = 0$, $f(1) = 1$.
2. A password consists of the digits `0` to `9` and the special symbols `*` and `#`. How many 6 to 8-symbol passwords are there if
   (a) all symbols must be different?
   (b) the password must have at least one digit?
   (c) all the digits in the password must be the same?
   (d) no two special symbols are consecutive? (**Hint:** Write a recurrence.)
3. Use the pigeonhole principle to prove the following propositions.
   (a) Among the 20000 students at CUHK, there are at least 1100 that all live in the same district. (Hong Kong has 18 distincts.)
   (b) You throw three six-sided dice repeatedly. The score of each throw is the sum of the face values of the three dice. Among 50 repetitions, at least four have the same score.
   (c) In every graph with at least two vertices there are two distinct vertices of equal degrees.
4. For each of the following pairs of functions, say whether (i) g is $o(f)$, (ii) g is $\Theta(f)$, or (iii) f is $o(g)$. Justify your answer.
   (a) $f(n) = e^n$, $g(n) = n^e$.
   (b) $f(n) = n^n$, $g(n) = 2^{n^2}$.
   (c) $f(n) = \sqrt{n}$, $g(n) = 2^{\sqrt{\log n}}$.
   (d) $f(n) = f(\lfloor n/5 \rfloor) + 2 \cdot f(\lfloor 2n/5 \rfloor) + n^2$, $g(n) = 1\log 1 + 2\log 2 + \cdots + n\log n$. ($\lfloor x \rfloor$ is the largest integer not exceeding x.)

5. DNA(*Deoxyribonucleic acid*) is a molecule that carries the genetic instructions for all known organisms and many viruses. It consists of a chain of bases. In DNA chain, there are four types of bases: **A**, **C**, **G**, **T**. For example, a DNA chain of length 10 can be **ACGTACGTAT**.

   (a) Let $g(n)$ be the number of configurations of a DNA chain of length n, in which no two **T** are consecutive and no two **G** are consecutive. Write a recurrence for $g(n)$.

   (b) Solve the recurrence from part (a).

   (c) Let $h(n)$ be the number of configurations of a DNA chain of length n, in which no two **T** are consecutive, no two **G** are consecutive, and **T**, **G** are not next to each other. Write a recurrence for $h(n)$.

   (d) Solve the recurrence from part (c).

6. A pair of permutations of $\{1, \ldots, n\}$ is a *special pair* if there is some position in which they differ by exactly one. For example, $\{(3,1,2,4),(1,4,3,2)\}$ (when $n = 4$) is a special pair because they differ by exactly one in the third position, but $\{(1,2,3,4),(1,4,3,2)\}$ is not a special pair. A set S_n of permutations of $\{1, \ldots, n\}$ is a *special set* if every two permutations within S_n are a special pair.

   (a) Show that when $n = 3$, there exists a special set of size 3 but no special set of size 4.

   (b) Show that if S_n is a special set, the function $f\colon S_n \to \{0,1\}^n$ given by $f((x_1, x_2, \ldots, x_n)) = (x_1 \bmod 2, x_2 \bmod 2, \ldots, x_n \bmod 2)$ is injective.

   (c) Use part (b) to show that if S_n is a special set then $|S_n| \leq 2^n$.

   (d) Define the sets $S_1, S_2, \ldots$ recursively by the formula $S_n = A_n \cup B_n \cup C_n$ where

$$\begin{aligned} A_n &= \{(n, n-1, x_1, x_2, \ldots, x_{n-2})\colon (x_1, x_2, \ldots, x_{n-2}) \in S_{n-2}\}, \\ B_n &= \{(x_1, n, n-1, x_2, \ldots, x_{n-2})\colon (x_1, x_2, \ldots, x_{n-2}) \in S_{n-2}\}, \\ C_n &= \{(n-1, x_1, n, x_2, \ldots, x_{n-2})\colon (x_1, x_2, \ldots, x_{n-2}) \in S_{n-2}\}. \end{aligned}$$

   with $S_1 = \{(1)\}$ and $S_2 = \{(1,2),(2,1)\}$. Show that S_n is a special set for all n.

   (e) Give a formula for the size of the sets S_n from part (d).

   (f) (**Extra credit**) For $n = 8$, can you find a special set larger than S_8 from part (d)?