

The number of text messages that Alice sends to Bob on a given day is a Geometric(Θ) random variable. Alice sent Bob 20 messages today. Assuming a Uniform(0, 1) prior on Θ what is Alice's posterior PDF?

random variable	PMF / PDF $f(t)$	range of t
Geometric(θ)	$(1 - \theta)^{t-1}\theta$	positive integers
Beta(α, β)	$\frac{(\alpha+\beta-1)!}{(\alpha-1)!(\beta-1)!}t^{\alpha-1}(1-t)^{\beta-1}$	$[0, 1]$

Solution: The conditional PDF of the number of messages X sent given Θ is Geometric(Θ), namely $f_{X|\Theta}(x|\theta) = (1 - \theta)^{x-1}\theta$, so the posterior PDF is

$$f_{\Theta|X}(\theta|x = 20) \propto f_{X|\Theta}(x = 20|\theta)f_{\Theta}(\theta) = (1 - \theta)^{19}\theta$$

for $0 \leq \theta \leq 1$. This is the PDF of a Beta(2, 20) random variable so the proportionality constant must equal $21!/19!1! = 21 \cdot 20 = 420$, namely

$$f_{\Theta|X}(\theta|x = 20) = 420(1 - \theta)^{19}\theta.$$