Practice questions

1. In an exam question half of the students scored 5 points, a quarter scored 3 points, and the rest scored no points. You are trying to figure out the average score by sampling two random students (with repetition) and asking for their score.
   (a) What is the PMF of the average score of the two sampled students?
   (b) What is the probability that the sample mean is equal to the actual mean?
   (c) What is the probability that the sample mean is within one point of the actual mean?

2. Let X_1, X_2, X_3 be independent samples of an Indicator(1/4) random variable. Calculate the PMF of the (a) sample mean (b) sample variance (c) sample standard deviation and (d) sample maximum.

3. A food processing company packages honey in glass jars. The volume of honey (in millilitres) in a random jar is a Normal$(\mu, 10)$ random variable for some unknown μ.
   (a) What is the PDF of the sample mean volume of six random jars?
   (b) What is the probability that the sample mean in part (a) is within 3 millilitres of the true mean μ?

4. Take $n = 100$ samples of an Indicator(0.01) random variable. Let $\overline{X}$ be the sample mean.
   (a) What is the probability that the sample mean $\overline{X}$ is within 0.005 of the true mean μ?
   (b) The Central Limit Theorem says that the event $\mu - \epsilon \le \overline{X} \le \mu + \epsilon$ should have similar probability to $-t \le N \le t$ for large n, a Normal$(0, 1)$ random variable N, and a suitable choice of t. What is the probability predicted for the event in part (a)?
   (c) How do the answers change if we lower the sampling error from 0.005 to 0.001?

Additional ESTR 2020 questions

5. If $\hat{X}$ is an unbiased estimator of a statistic μ, it is not true in general that $f(\hat{X})$ is an unbiased estimator of $f(\mu)$. In this question you will study the case when μ is the mean and $f(x) = x^2$.
   (a) Let $\overline{X}$ be the mean of n independent samples of some random variable with actual mean μ. Show that $\overline{X}^2$ is never an unbiased estimator of μ^2 (with one exception).
   (b) Show that the estimator $\frac{1}{n(n-1)} \sum_{i \neq j} X_i X_j$, where the sum is taken over all distinct, unordered pairs of samples is an unbiased estimator of μ^2.
   (c) Show that the estimator in part (b) is consistent. (**Hint:** Argue that the variance goes to zero as n becomes large.)
   (d) The estimator in part (b) is sensible only when at least two samples are available. Show that there is no general unbiased estimator of μ^2 if only one sample is available. (**Hint:** Assume the data consists of coin flips of unknown bias.)

6. The *support size* of a discrete random variable is the number of distinct values that it may take. How many samples do you need to calculate the support size of an unknown random variable, say with 50% probability, if you know (a) it takes at most K distinct values (b) it takes no value with probability less than $1/K$?