1. What is the minimum number of elements that need to be selected among the integers from 1 to 20 so that at least one pair of selected elements adds up to 21? Justify your answer.

   Solution: 11 are necessary because among the set $\{1, 2, \dots, 10\}$ every pair adds to at most 19. 11 are sufficient because if we partition the set into the 10 pairs $\{1, 20\}, \{2, 19\}, \dots, \{10, 11\}$, each of which adds to 21, by the pigeonhole principle among any choice of 11 numbers at least one of these pairs must be included.

2. Consider a connected graph whose sum of vertex degrees is 20. What is the largest possible number of vertices? Justify your answer.

   Solution: 11 again. 11 is possible because any tree on 11 vertices has 10 edges, so the sum of its vertex degrees is 20. 12 is not because any graph on 12 vertices with 10 edges must have at least two connected components (by Corollary 4 in Lecture 6).

3. How many length-10 strings of `0`s, `1`s, and `2`s are there that contain exactly two `0`s, three `1`s, and five `2`s? Show your calculations.

   Solution: $10!/(2!3!5!)$. The positions of the two zeros can be chosen in $\binom{10}{2}$ ways. Once these are fixed, the positions of the three ones can be chosen in $\binom{8}{3}$ ways. The positions of the twos are then determined. By the generalized product rule, the answer is $\binom{10}{2} \cdot \binom{8}{3} = 10!/(2!3!5!)$.

4. The number of length-n strings with symbols $\{\mathtt{A}, \mathtt{B}, \mathtt{C}\}$ in which no symbol appears consecutively three times (i.e., the patterns `AAA`, `BBB`, `CCC` are forbidden) is $\Theta(a^n)$. Find the number a.

   Solution: $1 + \sqrt{3}$. Let $f(n)$ be the number of such strings that start with a fixed symbol, say an `A`. If the second symbol is *not* an `A` then the remaining part can be any string of the same type that starts with a `B` or a `C`, so there are $2f(n-1)$ choices for it. If the second symbol is an `A` then the third symbol must be a `B` or a `C` and there are $2f(n-2)$ choices for the remaining part. Therefore f satisfies the recurrence $f(n) = 2f(n-1) + 2f(n-2)$.

   The solutions to this recurrence are linear combinations of x_1^n and x_2^n, where x_1, x_2 are the roots of $x^2 - 2x - 2 = 0$, namely $x_1 = 1 + \sqrt{3}$ and $x_2 = 1 - \sqrt{3}$. Therefore f is $\Theta((1 + \sqrt{3})^n)$. If we allow an arbitrary starting symbol there are three times as many choices and this doesn't affect the asymptotics.

5. A car number plate consists of two English letters from $\mathtt{A}, \mathtt{B}, \dots, \mathtt{Z}$ followed by three or four digits from $\mathtt{0}, \mathtt{1}, \dots, \mathtt{9}$. Assuming that all plates are equally likely, what is the probability that the letters are the same but all digits are different?

   Solution: Let A and B be there sets of 3-digit and 4-digit plates respectively, and A' and B' be the subsets of A and B in which the letters are the same and the digits are different. By the sum rule and the equally likely outcomes formula, the answer is $(|A'| + |B'|)/(|A| + |B|)$. The sets A

and B are product sets so $|A| = 26^2 \cdot 10^3$ and $|B| = 26^2 \cdot 10^4$. For A' and B' we use the generalized product rule to obtain $|A'| = 26 \cdot 10 \cdot 9 \cdot 8$ and $|B'| = 26 \cdot 10 \cdot 9 \cdot 8 \cdot 7$, so the desired probability is

$$\frac{26 \cdot 10 \cdot 9 \cdot 8 + 26 \cdot 10 \cdot 9 \cdot 8 \cdot 7}{26^2 \cdot 10^3 + 26^2 \cdot 10^4} = \frac{72}{3575}$$

which is about 2%.

6. You have a table with 23 rows, 8 columns, and an integer number in each cell. Show that there is a nonempty subset S of at most 4 rows so that if all rows not in S are deleted, the sum of the remaining numbers in every column is even.

   Solution: Let f be the function that takes a set R of rows of size 1 or 2, adds the rows in R, and replaces each entry by its value modulo 2. Then f has $23 + \binom{23}{2} = 276$ possible inputs and $2^8 = 256$ possible outputs. By the pigeonhole principle there must be two different sets R and R', each with at most two rows, so that $f(R) = f(R')$, namely their sums have the same value modulo 2. If R and R' happen to intersect this remains true even after we eliminate the common row from both, so we may assume they are disjoint. Then the sum of the rows in $R \cup R'$ must be even.

7. There are four boxes, two of which hide a prize, in random order. Alice points one of the boxes. Bob opens another box, shows that there is a prize in it, and takes it away. Alice can stay with her box, or switch to another one. If she wants the other prize, which option is better for her?

   Solution: Stay. If Alice stays, she has a 1/2 probability of winning a prize. For switching, the sample space consists of the outcomes (x, y, a, b, s), where x, y is the pair of boxes that hide the prizes, a is Alice's initial choice, b is the box revealed by Bob, and s is the box Alice switches to. The probabilities are described in a similar way as for the 3-box setup from Lecture 11. In particular, if Alice initially picks a box with a prize ($a = x$ or $a = y$) then Bob must pick the other ($b = y$ or $b = x$) and Alice never gets the prize ($s \neq x$ and $s \neq y$). The probability that Alice initially picks x or y is 1/2, so the probability that she wins by switching can be at most 1/2. (It is in fact strictly smaller because even if $a \neq x$ and $a \neq y$ Alice has some probability of losing, for example in the outcome $(x = A, y = B, a = C, b = B, s = D)$.)