Each question is worth 10 points. Please explain your solution clearly and concisely.

1. Is the following deduction rule valid?

$$\frac{\forall x\colon P(x) \text{ OR } Q(x)}{(\forall x\colon P(x)) \text{ OR } (\forall x\colon Q(x))}$$

2. Show that for every integer n, if $n^3 + n$ is divisible by 3 then $2n^3 + 1$ is *not* divisible by 3.

3. The vertices of graph G are the integers from 1 to 20. The edges of G are the pairs $\{x, y\}$ such that $\gcd(x, y) > 1$. How many connected components does G have?

4. What is $1 + (1 + 2) + (1 + 2 + 3) + \cdots + (1 + 2 + 3 + \cdots + 1000)$?

5. An $n \times n$ plot of land (n is a power of two) is split in two equal parts by a North-South fence. The Western half is sold and the Eastern half is split in two equal parts by an West-East fence. The same procedure is applied to the remaining $(n/2) \times (n/2)$ plots until 1×1 plots are obtained (see $n = 4$ example). How many units of fence are used?

6. A department has 10 men and 15 women. How many ways are there to form a committee with six members if it must have the same number of men and women?

7. A password is made of the digits `0`, `1`, ..., `9` and the special symbols `*` and `#`. The password must be 4-6 symbols long and contain at least one special symbol. How many passwords are there?

8. Show that every set of 10 integers, each of them between 0 and 25, contains two distinct subsets S, T *of the same size* such that the sum of the numbers in S equals the sum of the numbers in T.