Questions 1 to 6 are worth 10 points each. If you are in ENGG 2440A, please turn in solutions to *four* questions of your choice. If you are in ESTR 2004, please turn in solutions to *three* questions of your choice and the Mini-project (worth 30 points).

Please write your solutions clearly and concisely. If you do not explain your answer you will be given no credit. You are encouraged to collaborate on the homework, but you must write your own solutions and list your collaborators on your solution sheet. Copying someone's solution or pasting material you found online without credit will be considered plagiarism and may result in failing the whole course.

Questions

1. Can the following types of graphs exist?

   (a) A graph in which every vertex has degree at least two but has no cycle?

   (b) A graph with 3 connected components of 3, 4, and 5, vertices, respectively, and 8 edges?

   (c) A graph with 3 connected components of 3, 4, and 5, vertices, respectively, and 20 edges?

   (d) A graph with 6 vertices and 11 edges that is not connected?

2. Consider the following stable matching scenarios with n boys and n girls.

   (a) Give preference lists for $n = 3$ in which all of these matchings are stable:
   - Every boy gets his first choice and every girl gets her last choice;
   - Every girl gets her first choice and every boy gets his last choice;
   - Everyone gets their second choice.

   (b) Show that when $n = 3$, for any preference lists there exists at least one boy-girl matching that is *not* stable.

   (c) For every n show that if all the boys' preference lists are the same and all the girls' preference lists are the same, then there is exactly one stable matching.

3. Which of the following digraphs contain a (directed) cycle and which are acyclic?

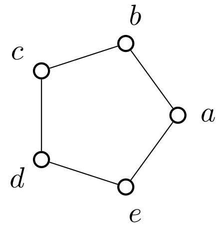
   (a) The vertices are all subsets of the set $\{1, \ldots, 100\}$. The edges are those pairs (S, T) such that the set difference $S - T$ has more than 50 elements.

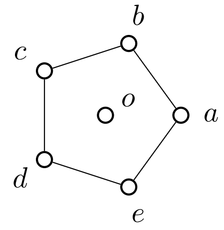
   (b) The vertices are the integers between 1 and 1000. The directed edges are those pairs (x, y) such that $y = x + 1$ if x is odd and $y = x/2$ if x is even.

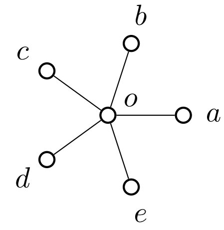
   (c) The vertices are the integers between 1 and 190. The edges are those pairs (x, y) such that 191 divides $x + 2y$.

4. The *square* of a graph G is a graph $\mathrm{sq}(G)$ with the same vertices such that $\{u, v\}$ is an edge in $\mathrm{sq}(G)$ if (and only if) there exists a path of length 2 from u to v in G.

   (a) Which of these graphs are squares of some (other) graph? Justify your answer.





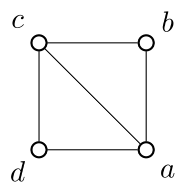


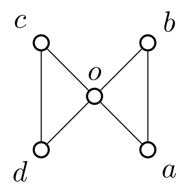
   (b) Show that if $\mathrm{sq}(G)$ is connected then G is connected but not bipartite.

   (c) Show that if G is connected but not bipartite than $\mathrm{sq}(G)$ is connected but not bipartite.

5. Let G be a connected graph. The spanning tree graph $\mathrm{Trees}(G)$ is defined as follows: The vertices of $\mathrm{Trees}(G)$ are the spanning trees of G. A pair of spanning trees $\{T, T'\}$ is an edge of $\mathrm{Trees}(G)$ if T and T' differ in exactly two edges (namely, there exist edges e of T and e' of T' such that the graphs obtained by removing e from T and e' from T' are the same).

   (a) Describe the spanning tree graphs of the following graphs:





   (b) Show that if G is a cycle of length n then $\mathrm{Trees}(G)$ is a complete graph on n vertices.

   (c) Show that for every G that is connected, $\mathrm{Trees}(G)$ is also connected.
   (**Hint:** Use induction on the number of edges in which two spanning trees differ.)

6. This question concerns spanning trees of the hypercube H_n which was defined in question 6 of Homework 3.

   (a) Give a spanning tree for H_3 that contains no path of length 6.

   (b) Show that H_n has a spanning tree that contains no path of length $2n$.
   (**Hint:** Use induction. You may need to prove a stronger proposition.)

   (c) Show that for all $n \geq 2$, every spanning tree of H_n contains a path of length $n + 1$.

   (d) **(Extra credit)** Does every spanning tree of H_n contain a path of length $2n - 1$?

ESTR 2004 mini-project Consider the following types of graphs on n vertices:

- A type A graph is the union of two edge-disjoint cycles, each of length n. (The two cycles cannot share any edges.)
- A type B graph is the union of two distinct cycles, each of length n. (The two cycles may share some edges but they are not identical.)

Let $s_{\mathrm{A}}(n)$ and $s_{\mathrm{B}}(n)$ be the maximum possible size of a shortest cycle among all type A graphs and type B graphs on n vertices, respectively.

(a) What is $s_{\mathrm{A}}(8)$ and $s_{\mathrm{B}}(8)$?

(b) Calculate the exact values of $s_{\mathrm{A}}(n)$ and $s_{\mathrm{B}}(n)$ for as many values of n as you can.

(c) Obtain the best upper and lower bounds you can for $s_{\mathrm{A}}(9,900)$ and $s_{\mathrm{B}}(9,900)$.