Questions 1 to 6 are worth 10 points each. If you are in ENGG 2440A, please turn in solutions to *four* questions of your choice. If you are in ESTR 2004, please turn in solutions to *three* questions of your choice and the Mini-project (worth 30 points).

Please write your solutions clearly and concisely. If you do not explain your answer you will be given no credit. You are encouraged to collaborate on the homework, but you must write your own solutions and list your collaborators on your solution sheet. Copying someone's solution or pasting material you found online without credit will be considered plagiarism and may result in failing the whole course.

Questions

1. For each pair of propositions, say whether they are logically equivalent. Justify your answer.

   (a) P
   $P \longrightarrow P$

   (b) P IFF Q
   (P AND Q) OR (NOT P AND NOT Q)

   (c) ($P \longrightarrow Q$) OR (NOT P AND Q)
   P XOR Q

   (d) If Alice goes then Bob will also go; if Alice and Bob both go, then Charlie won't go.
   If Alice doesn't go then Charlie will go.

   (e) Xiao will go only if Janice is going.
   If Janice goes then Xiao will go.

2. The following propositions are about a group of officers, some of whom are secret agents. $Agent(x)$ means that person x is a secret agent. $Knows(x, y)$ means that person x knows the status of person y (if y is a secret agent or not). Translate the following propositions into plain English.

   (a) $\exists x : \text{NOT } Knows(x, x)$

   (b) $\forall x, y : (Knows(x, y) \text{ AND NOT } Agent(x) \text{ AND } Agent(y)) \longrightarrow (x = \text{David AND } y = \text{Pele})$

   (c) $\forall x \exists y : Knows(x, y) \text{ AND } Agent(y)$

   (d) $\forall x : (Agent(x) \text{ AND } x \neq \text{Alice}) \longrightarrow \forall y : Knows(x, y)$

   (e) $\exists x \forall y, z : Agent(x) \text{ AND } Knows(y, x) \text{ AND } (z \neq x \longrightarrow \text{NOT } Knows(x, z))$

3. Translate the following English sentences into propositions using quantifiers and logical symbols. You may use $B(x, y)$ for "countries x and y border each other" and $L(x, y)$ for "country x is larger than country y." (Assume no two countries are of the same size.)

   (a) No country borders itself.

   (b) USA is neither the biggest nor the smallest country.

   (c) Norway does not border North Korea, but they border a country in common.

   (d) Australia is not the only country without borders.

   (e) Nauru is the smallest country without borders.

   (f) **(Extra credit)** Exactly two countries are larger than China.

4. Say which among the following pairs of propositions with quantifiers are logically equivalent. Justify your answer.
To say that two propositions are equivalent, describe their common meaning in English. To argue that two propositions are not equivalent, describe a possible world in which one of them is true and the other false. (For example, in a world where Alice is Rich but Bob is not, the proposition $\exists x\colon R(x)$ is true but $\forall x\colon R(x)$ is false, so the two cannot be equivalent.)

   (a) $\exists x \forall y : P(x, y)$
   $\exists x \forall y : P(y, x)$

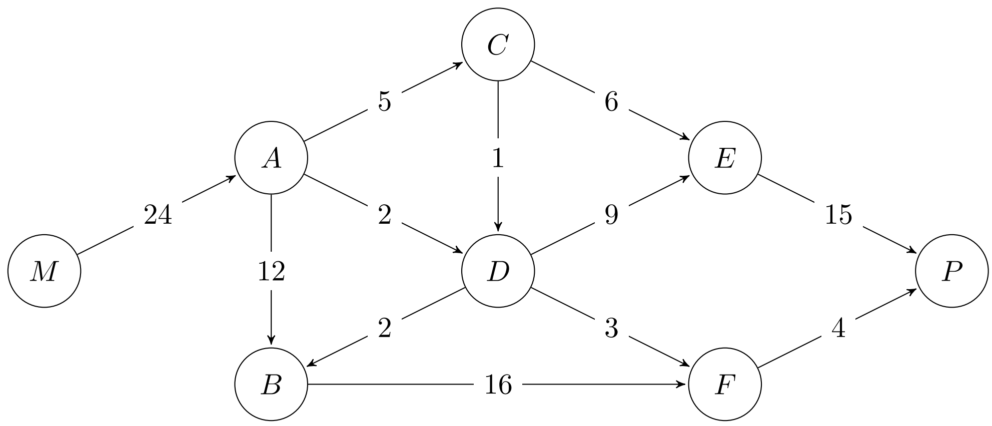
   (b) $(\forall x : P(x))$ AND $(\forall y : Q(y))$
   $\forall x, y : P(x)$ AND $Q(y)$

   (c) $(\forall x : P(x)) \longrightarrow (\forall x : Q(x))$
   $\forall x : P(x) \longrightarrow Q(x)$

   (d) $(\forall x : P(x)) \longrightarrow (\forall y : Q(y))$
   $\forall x, y : P(x) \longrightarrow Q(y)$

   (e) $(\exists x : P(x))$ AND (NOT $\forall x : P(x)$)
   $\exists x, y\colon P(x)$ XOR $P(y)$

5. You are organizing the distribution of durians around CUHK. You will hire a porter to get as many as possible from the University MTR station (M) to the Science Park (P). Here is the network of trails that can be used to carry the durians.



The number above each arrow is the largest number of durians that the porter is willing to move along that trail. The porter may make multiple trips, but is not allowed to move more than this many durians overall. For example, the porter can move a total of 6 but not 7 durians from C to E.

   (a) How should you instruct the porter to move the durians in order to maximise the number of them that reach the Science Park?

   (b) Explain convincingly why it is not possible for any extra durians to make it to the Science Park.

6. Express the following predicates about numbers (non-negative integers) using quantified formulas and the symbols 0, 1, =, +, ×, and E. The first five symbols have their usual meaning and $m\mathrm{E}n$ stands for m^n. For example, "$m = \sqrt{n} + 1$" can be expressed as $\exists r\colon m = r + 1$ AND $n = r \times r$.

   (a) i. $m \leq n$ and ii. $m < n$.

   (b) $m \bmod q = r$ meaning "r is the remainder when m is divided by q."

   A (binary) string is a finite sequence of `0`s and `1`s. We will represent the string x by the number whose binary expansion is $\mathtt{1}x$. For instance, string `0` is represented by number 2, string `110` is represented by number 14, and the empty string is represented by number 1.

   With this convention, predicates about strings can be expressed as predicated about numbers. For example, the predicate "the last bit of (string) x is `1`" is expressed by $x \bmod 2 = 1$. Express the following predicates:

   (c) $\mathrm{bit}(x, m)$ meaning "the m-th bit of string x is `1`".

   (d) $\mathrm{len}(x, m)$ meaning "string x has length m".

   (e) $z = x \circ y$ meaning "string z is the concatenation of strings x and y".
   (Concatenation means z is obtained by typing first x then y, e.g., $\mathtt{10011} = \mathtt{10} \circ \mathtt{011}$.)

   (f) **(Extra credit)** $m = 1 \times 2 \times \cdots \times n$.
   (**Hint:** Although this question is about *numbers*, strings may come in handy.)

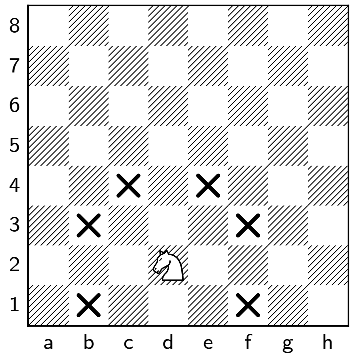
ESTR 2004 mini-project Call a chessboard configuration *good* if it satisfies the following constraints:

i. Each row and each column contains exactly one white knight and exactly one black knight.

ii. No two knights of the same color attack one another.

iii. Each knight attacks at least one knight of the opposite color.

A knight attacks all pieces that are two squares away in one direction and one square away in the other direction, as indicated by the crossed out squares in the following diagram:



(a) Write a propositional formula in conjunctive normal form representing the good configurations. Clearly explain the meaning of your formula.

(b) Use the computer to find good configurations for boards of size 6×6, 8×8, and 12×12 (if any).

(c) For which values of n do good $n \times n$ chessboard configurations exist?

(d) Now replace the constraints ii. and iii. by

ii'. No two knights of opposite colors attack one another.

iii'. Each knight attacks at least one knight of the same color.

and repeat parts (a), (b), and (c).