Practice questions

1. A point is chosen uniformly at random inside a triangle with base 1 and height 1. Let X be the distance from the point to the base of the triangle. Find the CDF and the PDF of X. (*Textbook problem 3.2.5*)

   Solution: The PDF of the point is uniform over the triangle which has area $1/2$, so it has value 2 inside the triangle and zero outside. The event $X > x$ consists of all the points in the triangle that are at distance more than x from the base, which is itself a triangle of base and height $1 - x$. Therefore $\mathrm{P}(X > x) = 2(1 - x)^2/2 = (1 - x)^2$. The CDF is $\mathrm{P}(X \le x) = 1 - (1 - x)^2 = 2x - x^2$ and the PDF is $f_X(x) = d\,\mathrm{P}(X \le x)/dx = 2(1 - x)$.

2. There are 100 students in class. The arrival times of students (in minutes) are exponential random variables with rate $\lambda = 0.2$, starting from 09:20.

   (a) What is the expected number of students that have arrived by 09:30?

   (b) Assuming students' arrivals are independent, what is the probability that everyone has made it by 09:45?

   Solution: Let T_i be the arrival time of student i. The CDF of T_i is $F_{T_i}(t) = 1 - e^{-\lambda t}$.

   (a) The probability that a given student has arrived by 09:30 is $P(T_i \le 10 \approx 0.865$. The number of students that have arrived by 09:30 is $X_1 + \cdots + X_{100}$ where X_i is an indicator random variable for the event $T_i \le 10$. By linearity of expectation the expected number of such students is the sum of $\mathrm{P}(T_i \le 10)$ as i ranges over the 100 students, which is about 86.5.

   (b) The probability that any given student has arrived by 09:45 is $p = P(T_i \le 25) \approx 0.993$. The number of students arriving before 09:45 is a Binomial$(100, p)$ random variables, so the probability they all arrived by this time is $p^{100} \approx 0.509$.

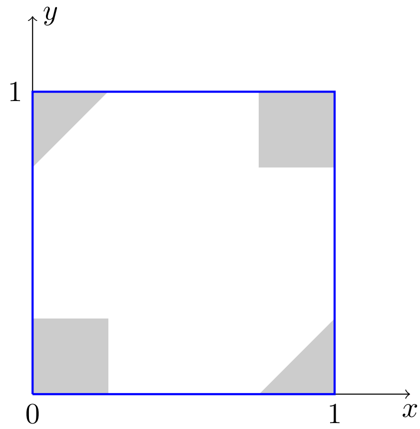
3. Three points are dropped at random on the perimeter of a circle with 1 unit circumference.

   (a) What is the probability that they all fall within $1/4$ of a unit of one another?

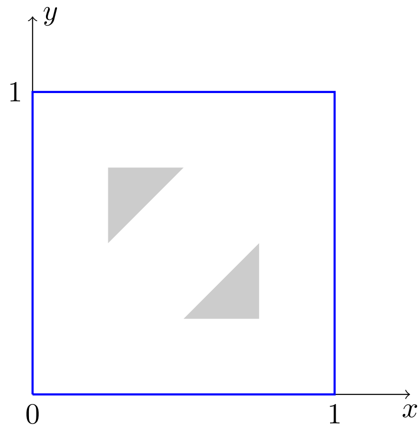
   (b) What is the probability that every pair of them is at least $1/4$ of a unit apart? (**Hint:** Fix one of the three points.)

   Solution: Let's call the three points a, b, and c. By symmetry, we can position a on the circle in an arbitrary way. Let X and Y be the positions of b and c relative to a clockwise along the circle. We model X and Y as independent Uniform$(0, 1)$ random variables.

   (a) The event E is the intersection of events A, B, C described by the predicates: (1) $x \in [0, 1/4] \cup [3/4, 1]$ (b is close to a); (2) $y \in [0, 1/4] \cup [3/4, 1]$ (c is close to a); and (3) $|x - y| \in [0, 1/4] \cup [3/4, 1]$ (b is close to c, clockwise or counterclockwise). $A \cap B \cap C$ is the shaded set in the following diagram and has probability $3/16$.



(b) The event E' of interest is now $A' \cap B' \cap C'$, where A', B', C' are the sets (1) $x \in [1/4, 3/4]$ (b is far from a); (2) $y \in [1/4, 3/4]$ (c is far from a); and (3) $|x - y| \in [1/4, 3/4]$ (b is far from c). This is represented by the shaded region below and has area $1/16$.



Another way to solve part (b) (or to check your answer) is via the axioms of probability. The complement of E' equals $A \cup B \cup C$ (some pair of points is close), so by inclusion-exclusion:

$$\begin{aligned} \mathrm{P}(E'^c) &= \mathrm{P}(A \cup B \cup C) \\ &= P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C). \end{aligned}$$

Here, A is the event that points a and b are less than $1/4$ of an inch apart, so $\mathrm{P}(A) = 1/2$. For the same reason $\mathrm{P}(B) = \mathrm{P}(C) = 1/2$. The events A, B are independent so $\mathrm{P}(A \cap B) = \mathrm{P}(A)\,\mathrm{P}(B) = 1/4$. For the same reason $\mathrm{P}(B \cap C) = \mathrm{P}(C \cap A) = 1/4$. In part (a) we calculated that $\mathrm{P}(A \cap B \cap C) = 3/16$, so

$$\mathrm{P}(E'^c) = 3 \times \frac{1}{2} - 3 \times \frac{1}{4} + \frac{3}{16} = \frac{15}{16}$$

and $\mathrm{P}(E') = 1/16$.

4. A coin has probability P of being heads, where P itself is a Uniform$(0, 1)$ random variable. Find the PMF of the number of heads after performing two independent coin flips.

Solution: Let N be the number of heads in two con flips. The conditional PMF of X given P is $f_{X|P}(x|p) = \binom{2}{x} p^x (1-p)^{2-x}$. As P is a Uniform$(0, 1)$ random variable, its PDF is $f_P(p) = 1$ when $0 \le p \le 1$ and 0 otherwise. By the total probability theorem, the (unconditional) PMF of X is

$$f_X(x) = \int_{-\infty}^{\infty} f_{X|P}(x|p) f_P(p) dp = \int_0^1 \binom{2}{x} p^x (1-p)^{2-x} dp.$$

It remains to calculate this integral for $x = 0, 1, 2$:

$$f_X(0) = \int_0^1 (1-p)^2 dp = \frac{1}{3} \qquad f_X(1) = \int_0^1 2p(1-p) dp = \frac{1}{3} \qquad f_X(2) = \int_0^1 p^2 dp = \frac{1}{3}.$$

5. Here is a way to solve Buffon's needle problem without calculus. Recall that an ℓ inch needle is dropped at random onto a lined sheet, where the lines are one inch apart.

   (a) Let A be the number of lines that the needle hits. Let B be the number of times that a polygon of perimeter ℓ hits a line. Show that $\mathrm{E}[A] = \mathrm{E}[B]$. (**Hint:** Use linearity of expectation.)

   (b) Assume that $\ell < \pi$. Calculate the expected number of times that a circle of perimeter ℓ hits a line.

   (c) Assume that $\ell < 1$. Use part (a) and (b) to derive a formula for the probability that the needle hits a line. (**Hint:** The number of hits is a Bernoulli random variable.)

Solution:

(a) Suppose the polygon has n edges of length $a_1, a_2, \ldots, a_n$. Break up the needle into segments of lengths $a_1, a_2, \ldots, a_n$. Let A_i and B_i be the number of lines hit by the i-th segment of the needle and the i-th edge of the polygon, respectively. Then

$$A = A_1 + \cdots + A_n \quad \text{and} \quad B = B_1 + \cdots + B_n.$$

By linearity of expectation

$$\mathrm{E}[A] = \mathrm{E}[A_1] + \cdots + \mathrm{E}[A_n] \quad \text{and} \quad \mathrm{E}[B] = \mathrm{E}[B_1] + \cdots + \mathrm{E}[B_n].$$

Since the i-th edge of the polygon and the i-th segment of the needle are identical, $\mathrm{E}[A_i] = \mathrm{E}[B_i]$. It follows that $\mathrm{E}[A] = \mathrm{E}[B]$.

(b) Let C be the number of times a circle intersects a line. We calculate the p.m.f. of C. Let d be the line segment representing the diameter of the circle that is perpendicular to the lines on the sheet. Since $\ell < \pi$, the length of d is less than 1. The circle hits a line twice if d crosses a line, once if d touches one of the lines, and zero times if d does not intersect any of the lines. The probability that d crosses a line is exactly the length of d, namely ℓ/π, and the probability that d touches a line is zero. Summarizing, the p.m.f. of C is

c	0	1	2
$\mathrm{P}(C = c)$	$1 - \ell/\pi$	0	ℓ/π

Therefore $\mathrm{E}[C] = 2\ell/\pi$.

(c) If we view the circle as a polygon with infinitely many sides, putting together part (a) and (b) we get that $\mathrm{E}[A] = \mathrm{E}[C] = 2\ell/\pi$. Since $\ell < 1$, the number of times the needle intersects a line is a 0/1 valued random variable, so $\mathrm{E}[A] = \mathrm{P}(A = 1) = \mathrm{P}(\text{the needle hits a line})$. Therefore the probability the needle hits a line is exactly $2\ell/\pi$.