A 6-faced die is rolled 15 times. Show that the probability of 6 distinct values occurring in some 6 consecutive rolls (e.g., $\texttt{13}\underline{\texttt{453261}}\texttt{64}\cdots$) is no more than 20%.

Solution: Let X_i be the indicator variable that the six consecutive tosses starting at position i are all distinct. Then $X = X_1 + X_2 + \cdots + X_{10}$ is the number of times six distinct consecutive tosses occur. By linearity of expectation,

$$\mathrm{E}[X] = \mathrm{E}[X_1] + \mathrm{E}[X_2] + \cdots + \mathrm{E}[X_{10}].$$

By the equally likely outcomes formula, $\mathrm{E}[X_i] = \mathrm{P}(X_i = 1) = 6!/6^6 = 5/324$, and $\mathrm{E}[X] = 10 \times 5/324 = 25/162$. The event that 6 distinct consecutive tosses occur somewhere is "$X \geq 1$". By Markov's inequality,

$$\mathrm{P}[X \geq 1] \leq \frac{\mathrm{E}[X]}{1} = \frac{25}{162} \approx 0.1543,$$

which is less than 20%.