Practice questions

1. Roll a 3-sided die twice. Let X be the larger number and Y be the smaller number you rolled. Find the joint PMF of X and Y, their marginal PMFs, and the expected value of $X + Y$.

   Solution: When $x > y$, the event $X = x$ and $Y = y$ can happen in 2 out of 9 possible ways: Either the first toss is x and the second toss is y, or the other way around. When $x = y$ then there is only one possible outcome. All other probabilities are zero. Summarizing, the joint PMF $p_{XY}(x, y)$ is

$x\backslash y$	1	2	3
1	1/9	0	0
2	2/9	1/9	0
3	2/9	2/9	1/9

   The marginal PMFs are obtained by adding the rows and columns, respectively:

z	1	2	3
$p_X(z)$	1/9	1/3	5/9
$p_Y(z)$	5/9	1/3	1/9

   The expected values are $\mathrm{E}[X] = 22/9$ and $\mathrm{E}[Y] = 14/9$. The expected sum is $\mathrm{E}[X + Y] = \mathrm{E}[X] + \mathrm{E}[Y] = 4$. Another way to see this is that if A and B are the first and second rolls then $X + Y = A + B$, so $\mathrm{E}[X + Y] = \mathrm{E}[A + B] = \mathrm{E}[A] + \mathrm{E}[B] = 2 \cdot (1 + 3)/2 = 4$.

2. On any given day between Monday and Saturday, the probability that you'll have a late snack is 20%, independent on the other days. You'll have a late snack on Sunday *if and only if* you didn't have one in any of the previous six days. What is the expected number of snacks you'll be having?

   Solution: Let X be the number of snacks you have between Monday and Saturday. Then X is a Binomial$(6, 0.2)$ random variable. The number of snacks you will be having in the whole week is

$$Y = \begin{cases} X, & \text{if } X > 0 \\ 1, & \text{if } X = 0. \end{cases}$$

   As Y is a function of X, we can use the formula for the expectation of the function of a random variable to obtain

$$\mathrm{E}[Y] = 1\,\mathrm{P}(X = 1) + 2\,\mathrm{P}(X = 2) + \cdots + 6\,\mathrm{P}(X = 6) + 1\,\mathrm{P}(X = 0).$$

   Using the binomial PMF formula $\mathrm{P}(X = x) = \binom{6}{x} \cdot 0.8^{6-x} \cdot 0.2^x$, we get that $\mathrm{E}[Y] \approx 1.46$.

3. Let p be a number between 0 and 1. Toss a p-biased coin. If the coin comes up heads, toss a fair coin and report the outcome twice (1 for heads, 0 for tails). If the coin comes up tails, report the outcomes of two independent fair coin tosses. Show that the marginal PMFs of your two reports are the same for every p, but the joint PMFs are all different.

Solution: Let X and Y be the two reports and E be the event that the biased coin comes out heads (which occurs with probability p). We calculate the joint PMF of X and Y using the total probability formula:

$$\mathrm{P}(X = x, Y = y) = \mathrm{P}(X = x, Y = y|E) \cdot p + \mathrm{P}(X = x, Y = y|E^c) \cdot (1 - p).$$

If E occurs, the event "$X = x$ and $Y = y$" can never occur if $x \neq y$. Otherwise, $X = 1, Y = 1$ and $X = 0, Y = 0$ both occur with probability half. If E does not occur then the events $X = x$ and $Y = y$ are conditionally independent so $\mathrm{P}(X = x, Y = y|E^c) = 1/4$, so

$$\mathrm{P}(X = x, Y = y) = \begin{cases} \frac{1}{2}p + \frac{1}{4}(1-p), & \text{if } x = y \\ \frac{1}{4}(1-p), & \text{if } x \neq y. \end{cases}$$

The joint PMF of X and Y is:

$x \backslash y$	0	1
0	$\frac{1+p}{4}$	$\frac{1-p}{4}$
1	$\frac{1-p}{4}$	$\frac{1+p}{4}$

and so the joint PMFs are all different as the value of p changes. On the other hand, the marginal PMFs, which equal the row/column sums, are uniform ($\mathrm{P}(X = 0) = \mathrm{P}(X = 1) = \mathrm{P}(Y = 0) = \mathrm{P}(Y = 1) = 1/2$) and therefore the same for all p.

4. Alice has two identical envelopes containing \$1 and \$2 respectively. She shows a random envelope to Bob, who guesses the amount of money in the envelope and then opens it. If Bob's guess is correct he collects the money in the envelope. If not, Bob gets nothing. Suppose Alice picks the \$1 envelope with probability p and the \$2 envelope with probability $1 - p$. Assuming Bob knows the value of p, how should Bob guess (as a function of p)?

   Solution: Bob should guess in a way that maximizes his expected utility. If Bob guesses \$1 his expected utility (in dollars) is $1 \cdot p + 0 \cdot (1 - p) = p$. If he guesses \$2 his expected utility is $0 \cdot p + 2 \cdot (1 - p) = 2(1 - p)$. Therefore Bob should guess \$1 when $p \geq 2(1 - p)$ and \$2 when $p \leq 2(1 - p)$. The equation $p = 2(1 - p)$ solves to $p = 2/3$, so Bob should guess \$1 when $p \geq 2/3$ and \$2 when $p \leq 2/3$. (When $p = 2/3$ both guesses achieve the same expected utility.)

5. 100 balls are tossed at random into 100 bin. Each ball is equally likely to land in any of the bins, independently of the other balls. What is the expected number of bins that receive exactly one ball?

   Solution: Let X_i be an indicator random variable for the event that the i-th bin receives exactly one ball ($X_i = 1$ if this happens, $X_i = 0$ if it doesn't.) Then the number of bins X with exactly one ball is

   $$X = X_1 + X_2 + \cdots + X_{100}.$$

   Even though the events $X_1 = 1, \ldots, X_{100} = 1$ are not independent, we can apply linearity of expectation to express $\mathrm{E}[X]$ as

   $$\mathrm{E}[X] = \mathrm{E}[X_1] + \cdots + \mathrm{E}[X_{100}].$$

   Since X_i is an indicator random variable, $\mathrm{E}[X_i] = \mathrm{P}(X_i = 1)$. The number of balls in bin i is a Binomial$(100, 1/100)$ random variable, so the probability that bin i has precisely one ball is

   $$\mathrm{P}(X_i = 1) = 100 \cdot (1/100) \cdot (99/100)^{99}$$

   and so $\mathrm{E}[X] = 100 \cdot 100 \cdot (1/100) \cdot (99/100)^{99} = 99^{99}/100^{98} \approx 36.973$.